Liability Models

Kevin C. Kaufhold
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Introduction

Portfolio theory will typically concentrate on the asset side of the investment process, with relatively little detail given to liabilities. The cash flow withdrawal stream of a mature portfolio is of vital concern however to any investor contemplating retirement. It should therefore be given a deal of great attention. Arguably, analysis of investment assets is simply incomplete without a corresponding analysis of the investment-related liabilities.

The prior chapter has clearly demonstrated that variance of pricing returns is vastly reduced in long-term time frames. While standard deviation loses much of its applicability as an appropriate measure of risk over the long-term, another type of risk becomes of great concern - that of a shortfall from one's investment objectives. As standard deviation decreases over long time frames, the distinct possibility of a shortfall in the funding of investment goals dramatically increases. This chapter focuses on this probability.

Safety First [1]

In an amazing case of parallel thought processes, A.D. Roy published an article in 1952 about the same time as Markowitz's original paper on portfolio selection.[2] Roy and Markowitz never heard of each other and never met, yet both papers followed an almost identical reasoning. The primary difference between the two was that Roy focused on the downside risk of a portfolio while Markowitz concentrated on the entire range of asset combinations. No one before had attempted a coherent theory of portfolio selection, and yet the two papers were published within months of each other. Roy's paper is the earliest known effort dealing with shortfall risk in investment analysis. Markowitz himself commented on some notions of semi-variance in his dissertation in 1959. Many years later, a shortfall risk constraint was developed for use with Markowit'z means-variance methods, and a risk measure based on lower confidence interval limits was also attempted.[3]

Roy's safety-first principle states that the optimal portfolio should minimize the probability that portfolio return will fall below a designated threshold level. The probability of shortfall is the expected return of the portfolio minus the threshold rate of return. The equation can be standardized to produce the Safety First Ratio (SFR). Then, the probability that the return will be less than the SFR can be ascertained by simply referring to a standard normal cumulative density function.[4]

$$SFR = \min p\ (r_p < r_L)$$
$$SFR = [\ E(r_p) - r_L] / \sigma_p$$
$$p\ (r_p < r_L) \text{ is: } N\ (-\ SFR).$$

Where, SFR is the shortfall ratio, and r_L is the threshold return level. An asset allocation with the lowest probability of shortfall should ultimately be chosen. This will give the lowest probability of a portfolio not meeting the threshold return rate.

Kataoka uses a similar safety first criterion: max r_L, s.t. p (rp $<r_L$) $<= \alpha$, where α is a predetermined probability value (such as .005). Telser's safety-first concept was: max r_p, s.t. p ($r_p < r_L$) $<= \alpha$. [5]

The Sharpe Ratio is close to the safety-first ratio.[6] Using the risk free rate of return instead of the threshold portfolio return level produces the Sharpe Ratio, which is: $S = (r_p - r_f) / \sigma_p$. The portfolio that maximizes the Sharpe Ratio will also minimize the probability that the portfolio return will be less than risk free rate of return.

One aspect of the above analysis is striking. Equity allocations that are considered optimal for means-variance purposes may be lower or higher than recommended allocations when considering safety-first criterion. An optimal portfolio is not necessarily desired, only the allocation of assets that minimizes the probability of shortfall from achieving a designated threshold rate. The portfolio that maximizes the Roy criterion, however, will at least lie on the efficient frontier in mean-variance space.

Below Target Risk Measures. Several papers have focused on a specific minimum target value of a portfolio. By identifying a "doomsday" rate of return that must be achieved in order to fund liabilities, the risks of shortfall can be statistically derived. The effort is essentially an updated version of Roy's concepts. Below target probability (BTP) and below target risk (BTR) are statistical measurements of the risks of shortfall. BTR penalizes large shortfalls more than some shortages. It is based on an actuarially required threshold rate that is necessary to fund the anticipated liabilities. BTP also develops a target rate but does not weight large deviations below target heavier than small ones. These risk measures are ideally suited to work in conjunction with asset-liability modeling. [7]

Summary of Safety-First. While Roy's method looks at risk as the probability of not meeting target objectives and thus heads towards shortfall considerations, it does have serious disadvantages as a practical tool. SFR is still based on standard deviation, which becomes rather useless in the long-term. Most significantly, it does not contain any modeling of liabilities, only an assumed and targeted threshold level. For extensive modeling of shortfalls across varying holding periods, one needs to more fully develop the liabilities and planned withdrawals from a portfolio.

Downside Risk from a Benchmark [8]

Where the primary investment goal is to meet or exceed a benchmark index, various statistical measures have been developed over the years to estimate the possibility of missing the benchmark. Standard deviation and variance measures are both symmetric in nature and therefore assume a normal distribution. Where an asymmetric return develops, standard deviation would not necessarily be an adequate measure of risk, since the distribution may be skewed towards (or away from) the level of the benchmark return.

Tracking Error. To measure the risk of a downside loss, a benchmark of risky assets must first be selected, and then the risk of not meeting the benchmark can be evaluated. This by itself may introduce error into the calculations, where the investment does not align its performance to that of the benchmark. Tracking error is the standard deviation of the difference in return between the investment and a target benchmark. Since standard deviation is involved in the estimation of tracking error, there is an assumption of normality in the estimation. Thus, tracking error may not identify the complete probability of the downside where the distribution is skewed.

Shortfall probability is the total probability of returns falling below the benchmark return level. A simple but effective way to use this measure is to compare the downside shortfall probability against a default free rate of return (Treasury bills or government notes). It is also common to evaluate shortfall probability against other benchmarks with higher rates of return (such as equity or balanced fund benchmarks). This measure does not account for the severity of the downside risk, however.

Expected Shortfall is the average distance of downside returns relative to the risky benchmark return. This method does not factor in the variability and distribution of downside returns, and thus suffers from the same problem regarding asymmetry as does standard deviation.

Semivariance is the variance of only the downside returns relative to the benchmark. It measures only that portion of the total probability distribution function of returns that is below the benchmark return rate. This measure is most consistent with many long-term investors' notion of risk being the probability of a loss. It is therefore often considered a preferred measure of downside risk relative to a benchmark. Semivarance corrects for problems of asymmetry in returns, since it only measures the returns falling below the benchmark return. If a skew develops below the benchmark return, semivariance will statistically account for that. Outliers are also more appropriately dealt with, since semivariance squares the distance from each observation below the benchmark return, thereby penalizing such observations more than with shortfall or expected shortfall probabilities.

In general, the concept of downside risk off of a benchmark has some advantages, especially with semivariance calculations capturing only the statistical downside of a probability distribution. Aggregate market-level benchmarks are still employed, however. Because these benchmarks reflect pricing behavior of capital markets, this form of downside risk suffers from a lack of consideration being given to the anticipated liabilities of particular portfolios.

Cost of Insuring Against Shortfall Risk

The costs of insuring against shortfall can be simulated with a put option of equal maturity to the investment horizon and with a strike price set at the forward of the underlying portfolio.[9] This is known as the Bodie Put, named after the developer of the concept. If equities were less risky in the long run than bonds, then the cost of insurance should decrease over time. Empirical results however, show that the put price insuring against the shortfall increases as the investment horizon lengthens.[10] Arguably then, standard deviation reductions in longer time frames are merely the result of statistical illusions, representing no loss of risk.[11] Advocates of insuring against shortfall risk have thus called for the design of a new line of financial products to ensure against severe market declines.

One paper suggests that Bodie's option is only measuring the increase in a stock's cumulative volatility over time, with the asset's return dispersion causing the option value to increase over time.[12] There would then be nothing inconsistent about decreasing pricing volatility over time versus increasing option value over the same time.

Option types of insurance to avoid shortfall contains some interesting and useful concepts, but the area is so theoretical and complex that it may be difficult to translate into practical applications. Further, the broader area of portfolio insurance was thought to play a role in the disastrous market freefall of October, 1987. There may be a general reluctance among investors and professionals alike to commit assets to the concept, as a result. Many long-term investors simply will not engage in extensive use of options to avoid the risk of shortfall, due to their "long-only" investing styles and beliefs.

Asset-Liability Modeling (ALM) [13]

Popularized in the 1970's, the modeling of shortfall by comparing asset and liability streams has become a standard technique in determining whether investment objectives are being fully met and funded. Some economic and financial observers feel the method has been developed *ad hoc* in nature, without a strong theoretical basis to it. Asset-liability projections are considered so important however, that an actuarial estimation of funding levels is mandated in many fiduciary settings, including ERISA. The method can be used for any investment portfolio having both asset contributions across time horizons and anticipated withdrawals at some point in time. Indeed, the Social Security Administration extensively utilizes actuarial models in evaluating future benefit and contribution levels.

With ALM, the stream of future assets and liabilities are estimated by bringing the asset and liability cash flow streams in forward time frames back to present cash value. Cash flows in the first forward ten years will be especially important to analyze if withdrawals or disbursements are to be engaged in. The identification and segmentation of liabilities is also very important to do, as this ascertains the benefit / liability stream that is absolutely required to be made. Classifications include mandatory versus discretionary payments, vested versus non-vested participants, and active versus retired lives. The general procedure calls for matching liabilities in various time frames with assets of equal or greater value in the same time frame.

The Definition of Risk. The overall objectives of the modeling process include:

1) Fund liabilities at the lowest cost;
2) Fund liabilities at the lowest risk;
3) Improve or enhance the financial statements of portfolio owner / plan sponsor;
4) Improve or enhance the credit rating of the portfolio owner / plan sponsor.

Risk with asset liability modeling is considered to be the failure to meet any of these four objectives, although the primary risk is commonly viewed as the failure to fully fund investment / plan objectives. If assets match liabilities, the plan has no risk associated with it. If assets generally cover the liabilities but without complete matching of time frames, there may be low levels of risk to the portfolio. If assets simply fail to cover anticipated liabilities, the portfolio or plan has high risks of underfunding or shortfall. Variance of the portfolio rate of return is not seen as a measure of risk by itself. Instead, the volatility of portfolio returns is considered one variable necessary to calculate the probability of not meeting plan objectives and / or to fully fund the objectives.

Types of Models / Procedural Steps. Several versions of ALM exist. Portfolio immunization (more extensively discussed below) concentrates on limiting interest rate sensitivity. Deterministic modeling provides one actuarially based answer of funding status for any given set of assumptions. This is a common method to use, but produces a single point estimate of whether investment objectives are fully funded. Stochastic projections explore the probability of results for the given input assumptions, thereby generating a range of values. While being less precise in nature, a probabilistic answer of adequate funding levels could be far more useful to portfolio owners / plan administrators, especially with so many of the actuarial assumptions being highly variable and uncertain in nature. Simulations of various scenarios are also quite useful and relevant, particularly when the probabilities of input assumptions as well as output results are factored into the process. Many asset-liability models focus on cash flow coverage, while other models concentrate on investment risks. Typically, if cash flow is an issue, the modeling emphasizes the matching of cash flows. If the portfolio is at or near a fully funded status and has no real difficulty of meeting short-term cash flow needs, the plan will have many of the trappings of a normal long-term investment portfolio, and the focus will be on business, financial, and market-level investment risks of plan assets.

The procedural steps of the asset-liability process could be summarized as:[14]

1. Document the amounts and time frames of the cash flows coming into and out of the portfolio or plan.

2. Define actuarial assumptions and develop a present cash value actuarial evaluation of the cash flows.

3. Simulate the financial conditions of the portfolio or plan under various scenarios.

4. Ensure a match between the asset duration and the liability duration for each time frame that the assets and liabilities are being held in.

5. Use US government, US agency and zero coupon bonds for at least the first ten years of the portfolio's liability schedule, in order to adequate match the near-term liabilities and avoid impacts from interest rate fluctuations.

6. Use MVO analysis on assets matched for time frames longer than ten years and / or for assets not devoted to cover plan liabilities (i.e. surplus assets). This will optimize the asset allocations held in longer time frames.

7. A full evaluation of the portfolio should identify the following:

    a. The potential to match the assets and liabilities as far into the future as possible.

    b. The extent and time frames of the funded and unfunded liabilities, on a present cash value basis.

    c. The identification of vested versus non-vested liabilities (or mandatory vs discretionary liability for endowments and foundations; or active, retired, and total lives).

    d. Projections of contribution levels necessary to fund investment objectives and account for other expenses to the portfolio or investment plan.

Actuarial Assumptions. Many demographic assumptions have to be made in the modeling process. The number of present and future participants and their categories; dates to retirement, disability or death and the mortality tables used to establish those estimated dates; hours worked of participants; termination of participants from the plan; and marital status are but a few of the many necessary inputs to the model. As a result of the Pension Protection Act of 2006 (PPA), each assumption in both single and multi employer plans must reasonably anticipate expected experience.[15]

One of the most important actuarial assumptions with asset-liability modeling is the estimated return on assets (ROA) of accumulated assets in the portfolio or investment plan. Assuming too high of a projected portfolio or plan return on assets will over-estimate the forward asset stream, potentially resulting in a disastrous underfunding of investment objectives when the returns fail to materialize. Estimating too low of a return could result in significant amounts of contributions or cash flows into the portfolio or plan that could be used elsewhere for business purposes or personal consumption. It is quite common for pension funds to use an assumed annualized rate of return of 7.5% on their investments, with adjustments occurring for varying asset allocations.

A problem also arises when there is an unexpected and dramatic shift in analyst forward return projections. For example, in the late 1990's, many professional financial sources were projecting relatively high forward rates of return. Actuaries had an abundance of such estimations to rely upon, in fact. After the stock market bubble burst in March, 2000 however, many analysts made an abrupt and wholesale shift in their forward estimations of many asset classes. This signaled the start of an era of underperformance, which will conceivably continue until such time as current asset valuations realign with long-term trends. Any portfolio or plan that relied upon late 1990's projections was caught in the economic upheaval of the times. To compound this problem, the IRS had a rule in place until 2002 requiring a maximum asset retainage of 125% of full funding levels. This had the practical effect of forcing defined benefit plans to increase benefit levels so as to avoid an overfunding of plan assets, and to do so at a time when many asset valuations were at all-time highs. DOL and ERISA anti-cutback rules then prohibited the funds from eliminating the newly granted benefits. Then, when forward rates of return dramatically dropped after 2000 and asset valuation levels returned to much lower price to value levels, the result was sudden and large actuarial underfunding of liabilities in numerous plans.

Also related to the ROA assumption is the methodology used to calculate asset values. In some circumstances, valuations can be either marked to market value or averaged over the last five years (this is commonly known as asset smoothing or actuarial value). SFAS 87 allows either asset smoothing or fair market value, while IAS 19 and FRS 17 requires only fair market value. Smoothing could either help or hurt an underfunded situation, depending upon the circumstances. If a market drop occurs over the course of a relatively short time-span, say one year, asset smoothing could assist the plan from being actuarially underfunded. This is because asset values would be based on a five-year moving average instead of being marked to market when the aggregate markets would have lowered price to valuation ratios. However, if the market drop continues for a more extended time period, (for example, three years of large negative returns, a fourth year of modest positive returns, and then a flat fifth year), asset smoothing could work against the funding status of the portfolio. By the time that asset values are sizably reduced in value through a smoothed average, the viability of the plan could be in grave doubt. This situation would continue even after the markets have once again started an upward trend, since asset value would be based on a moving average that still would be much lower than current market levels.

Then also factor in a reduced estimated forward rate of return, and funding problems quickly mount. Not only are asset values averaged or smoothed below current market levels, but the forward ROA assumption is also much lower. The combination of smoothing and a lower ROA assumption makes it very difficult for a plan to overcome projected underfunding without significant additional contributions being made. This very scenario has occurred among many defined benefit plans, since their smoothed asset valuation levels have suffered greatly from March, 2000 onward. Projected ROA's are much lower as well, reflecting reduced near-term expectations among many analysts. It all adds up to a very troubling set of circumstances.

Another very important assumption is the return on liability (ROL) estimate. This is essentially the discount rate used to calculate the present value of the liability stream. The higher the discount rate of the liability stream, the lower the present value of the liabilities. Using an inappropriate discount rate can produce severely distorted estimations of the liability stream.

As to interest rates and discount estimations, FAS 87, Paragraph 199 suggests that for pension benefit obligations, individual discount rates corresponding to the various benefit deferral periods should be selected. The purpose of this rule is use interest rates corresponding to the same time periods as the portfolio's / plans liabilities. Of course, the same procedure could be used whenever ALM is used, not just for pension benefit purposes. Para. 44 of FAS 87 allows employers of defined pensions to use rates of returns on high-quality fixed income investments available for the period to maturity of the benefits. FAS 106, Para. 186 provides for the measurement of a single amount that, if invested in a portfolio of high-quality debt instruments, would provide the necessary cash flows to pay the accumulated benefits when due. This is considered to be a high-quality zero coupon bond whose maturity dates and amounts would be the same as the timing and amount of the expected future benefit payment. Thus, setting the discount rate to a high quality zero coupon rate of similar duration and amount as the benefit (or liability) payment would be appropriate. This would conceivably generate several different discount rates, one for each time frame that liabilities are in.

On a cautionary note for publicly traded corporations having pension obligations, the SEC Guidelines call for action by the SEC Enforcement Division on estimated rates that cannot be justified or are considered too high. Restatement of financial statements, civil or criminal penalties could result. [16]

Inflation should also be factored into the analysis. If COLA's or anticipated inflation apply to the portfolio or plan, then inflation may be one the most significant forward liabilities. The accelerating cost of benefits must be carefully modeled in such instances, and various inflationary estimates should be evaluated for impact. Inflation and interest rate assumptions are considered key assumptions affecting both the asset and liability streams in the ALM modeling process.[17]

<u>ALM Equations.</u> The following equations and terms are used in asset-liability modeling and funding determinations. [18]

Assets = A

Liabilities = L

C = Contributions

Normal cost = NC

Unfunded Actuarial Liability = UAL

UVBL = Unfunded vested benefit liability

PVVB = Withdrawal Liability, or present value of vested accrued benefits

PVAB = present value of non-vested accrued benefits

PVB = Present value of benefits

Minimum Funding Requirements = NC+ Amort. of UAL – Credit Balance

Maximum Funding Requirements = NC + 10 year amort. of UAL or:
= NC + unfunded current liability (if greater)

Unfunded Period = Unfunded liability + (C – NC – Admin Exp)

Funding Ratio = A / L, or actuarial A / actuarial L

Withdrawal Liability (UVBL) = PVVB - A

Benefit Margin = C + NC – Admin Exp

Unfunded Liability = Actuarial L + Actuarial A

Contribution Margin (i.e. shortfall) = C – (NC + Admin Exp + amort. of UAL over selected period).

Credit Balance = total cumulative C – minimum required C

Portfolio immunization is a very useful tool with ALM, since the attempt is made to reduce interest rate sensitivity in a portfolio or plan by keeping the liabilities and the assets designed to cover those liabilities in the same holding period.[19] Any fluctuations in interest rates will then equally impact both the asset and liability streams, thereby "immunizing" the portfolio from adverse impacts of changes in interest rates. As stated in FAS 106, Para. 186, the market value of the liabilities should "equal the current market value of a portfolio of high-quality zero coupon bonds whose maturity date and amounts would be the same as the timing and amount of the expected future benefit payments." The general goal is to keep assets and liabilities moving in tandem as to valuation levels in spite of interest rate fluctuations.

There are two basic applications. The portfolio could be immunized against interest rate fluctuations at a market level, to achieve a target rate of portfolio return (this is market immunization). Or there could be a plan immunization, with a portfolio rate of return selected to cover future liabilities. With either application, the portfolio could be totally immunized (which matches the term of every liability with the term of an asset of equal present cash value), or partially immunized (where only the ins and out flows of cash are matched). Partial efforts are less expensive and more flexible than total immunization requirements since the entire portfolio does not have to be strictly structured. With partial immunization, the assets themselves do not have to be matched with the duration of the liabilities, only the cash flows of the assets and liabilities need to be matched. This is accomplished by filling shortfalls in cash flows with temporary borrowings. But, the overall shortfall risk of the portfolio can increase due to extra debt (i.e. liabilities) now being made part of the portfolio in order to cover the cash flow shortage.

Immunization can become quite sophisticated in its modeling. A variety of different interest rate environments should be simulated to ascertain the extent of cash flow or portfolio sensitivity. Multiple valuation methods and types should also be simulated to identify the duration and required hurdle rates of return for liabilities in every time horizon, given the different asset types and valuation calculations. The process can even involve stress tests and other techniques commonly used by banks and financial institutions. The general process contains similarities to Value at Risk methodologies as well as Monte Carlo simulation methods (both discussed below).

In spite of its advantages, Immunization may not be appropriate or possible in all situations. It may chiefly apply to certain segments of a portfolio or plan or certain time frames. While Immunization may assist in reducing the variability of the cash flows and funding status, it can constrain future investment policy, especially where total immunization is desired. It may also force the portfolio to heavily lean towards fixed income devices, because of the need to match near-term cash flows. Non-traditional assets are difficult to use for matching purposes in short and near-term time frames, because of their long lead-time for full realization of value. In fact, venture and real estate projects may themselves require cash injections during the early phases of the venture. Thus, equities and alternative assets will be relegated to long-term matches, leaving the portfolio to greatly tilt towards bonds to meet short-term cash demands.

<u>Personalized Indexes.</u> With so many items of the process being driven by factors at the portfolio or plan level, it is common to generate customized or personalized indexes that are uniquely tailored to the individual portfolio or plan. Market value, yield rates, duration of the investment devices, weights and returns are individualized for each portfolio, and a present value estimation is ultimately arrived at to measure the characteristics of both the asset and liability streams. A personalized index thus obviates the need for a single discount rate. It also avoids usage of a generic, market-level benchmark that may represent broader aggregate markets more than individual, investor-level shortfall probabilities. By moving away from an aggregate index and establishing an investor-level index, the liability schedule will be easier to ascertain and maintain, especially in light of individualized portfolio or plan objectives.

A customized index is better able to develop an investor-level funding target and establish a benchmark for asset and performance management. Asset allocations are largely based on the extent and duration of the liability stream. Typically, near-term liabilities are matched to assets also held in short to near-term time frames. These assets are allocated to bonds and cash equivalents. Then, assets designated for longer-time frames will be allocated to equities to cover the non-vested payments or active lives benefits / liabilities projected well into the future.

Two different portfolios or plans containing the same amount of assets could therefore have remarkably different asset allocations, rates of return, and variance of portfolio returns. Yet both portfolios would be considered "optimized" and having no substantial risk so long as the present cash value of the assets matches the amount and duration of the present cash value of the projected liabilities. The goal is not to meet or beat an aggregate benchmark, but to sufficiently fund liabilities of a particular investment plan.

Portfolio assets that are beyond the level necessary to fully match all known liabilities would be considered "surplus" and would not have to respond to an individualized, custom index. Instead, the surplus assets could be segregated into a distinct sub-portfolio that is responsive to an aggregate level benchmark. Allocations on surplus cash could be accomplished via traditional MVO procedures, since no duration or amount matching would be necessary with these assets.

<u>Handling Fully Funded and Underfunded Plans.</u> If investment objectives are at or near fully funded status, a strategy known as Surplus Optimization is used to decrease the probability of a future deficit. Consistent with principles of immunization, "optimization" occurs by matching the amount and duration of assets and liabilities being held in the same time frame. For example, if there is a current year income need, then assets are held in short-term times frames (typically fixed income short-term bonds) in sufficient amounts to fund that income need. Conversely, if there are no current income withdrawals but there is a projection of withdrawal in ten years, then assets should be held in a ten-year time frame in sufficient amounts to fund that liability. Allocation levels for the ten-year assets can also be optimized through MVO processes. The portfolio will not suffer from interest rate sensitivity since both assets and liabilities held in each time frame are equally affected by any change in rates. The rate of return for assets held in each time frame would be maximized consistent with asset allocations generated through MVO calculations.

If the investment objectives are not fully funded, then a Deficit Reduction Strategy is employed. Basically, the portfolio or plan tries to buy some time while things are brought back to a fully funded level. Often, the problem can be solved over time, with near-term problems being immediately addressed to maintain sufficient cash flow for liabilities, while longer-term problems are handled through incremental adjustments in funding or benefit / liability levels. Indeed, the real value of actuarial studies lies in the he ability to anticipate deficits far in advance of the projected date of deficit. This allows time to remedy the situation through a series of smaller structural changes to the portfolio or plan that can occur over longer time frames.

Increasing the expected returns of investments may be the quickest way to resolve underfunding issues, but this generally entails the assumption of higher pricing volatility risk since equities, international investments, and even non-traditional assets will be relied upon to generate the higher returns. For example, for current and near–term deficits, either or both the fixed income duration can be extended or allocations to non-income growth vehicles can be increased. Either method will lower the immunization of the portfolio since the asset durations are extended beyond the duration of the near-term liabilities. This also increases the statistical probability of shortfall in near-term time frames if the portfolio variance of returns goes up (which is likely to occur with a shift in allocation towards equities and other investments). If money is available for new contributions, the assets could be placed into short-term time frames to match the liabilities, resulting in no loss of immunization or increase in portfolio standard deviation.[20] For deficits projected to occur in longer time frames without any near-term needs pressing upon the portfolio, the problem can be dealt with by increasing the duration of assets to match the longer-term liabilities. This again involves allocating more assets to equities and non-traditional asset classes, thus placing assets with higher expected returns into the same time periods as the projected liabilities. If the present value of assets having higher expected rates of return are insufficient to cover the long-term liabilities, then new contributions in the longer holding periods would still be necessary.

Also to consider for an underfunded portfolio or plan, one could seek to reduce administrative expenses and anticipated future liabilities. The modeling process should identify the liabilities that are vested versus non-vested (or mandatory vs discretionary; or active versus retired lives). For endowments, some institutionals, and individual portfolios, near-term cash withdrawals could be varied to the extent of non-vested or non-mandatory liabilities. In a defined benefit ERISA context however, reductions in benefits are subject to the anti-cutback rules, so active lives versus retired lives should be utilized in the evaluation process. The ability to reduce benefits is severely limited by the anti-cutback provisions for retired lives, while there may be more of a possibility to alter future benefits for active lives that are not yet protected by anti-cutback statutory provisions.

In summary of asset-liability modeling, the process has much to offer since it incorporates liabilities into the modeling of shortfall risk. Over the years, ALM has matured to the point where probabilistic forecasting and simulations currently allow an investor the ability to determine whether shortfall of investment objectives is likely to occur in future time horizons. It also provides for personalized indexes, thus shifting the analysis away from an exclusive reliance of aggregate level risk and return definitions, moving instead towards investor-level risk factors. A large drawback with ALM however is the critical nature of the actuarial assumptions and procedures used. If any assumption is off of target, output of the model could prove to be seriously flawed. By moving to scenario analysis of various assumptions, this drawback can at least be minimized. Simulations add considerably to the value of the ALM process. Another disadvantage with ALM is in the somewhat artificial nature of funding calculations. With many statutory and regulatory requirements impacting ALM calculations, a "fully funded" status may be more reflective of definitions and procedures mandated by law and accounting methods than generated by economic circumstances. Still, asset-liability modeling is a commonly accepted and widely used procedure that provides a solid estimation of the probabilities of shortfall of investment objectives.

Cumulative Net Wealth [21] Cumulative wealth can be modeled by calculating end period wealth and compound rates of return. This is done by:

$$W_n = W_0\,(1+r_1)(1+r_2)\ldots(1+r_n)$$
$$r_g = (W_n / W_0)^{1/n} - 1$$

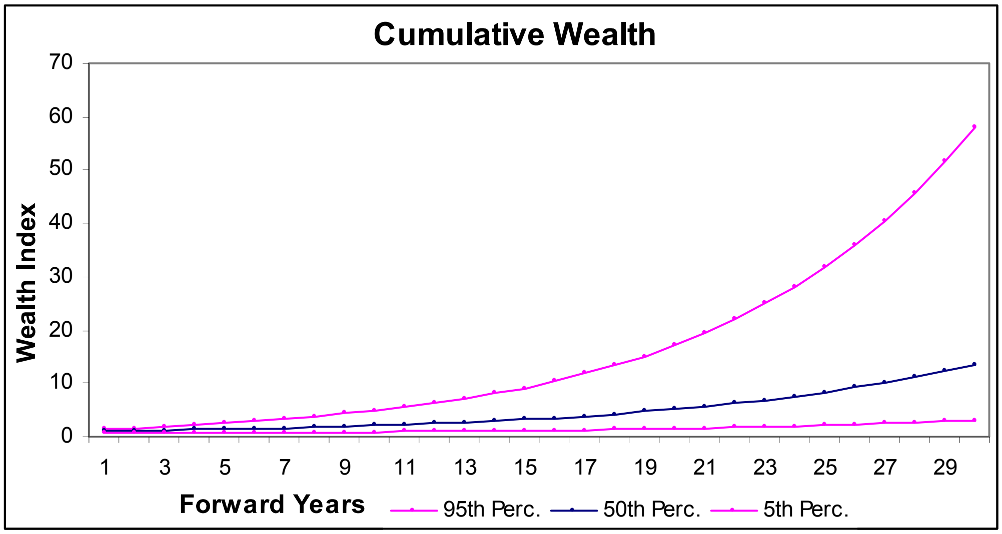
Where W_n is wealth in period n, W_0 is wealth in period 0, r_1 is return in period 1, r_g is the geometric rate of return, and n is the number of periods. Lognormal probability distributions are assumed for many forecasting models, due to returns of many assets being lognormally distributed. With many small losses and a few large gains as well the possibility that returns cannot fall below 100% negative return, a lognormal better describes asset returns than does a normal distribution, at least for some forecasting purposes. Even though optimized equity portfolios using historical data closely follows a normal distribution (Kaufhold, 2006:2), if a normal were used for forecasting work, returns could be infinitely small as well as infinitely large. A lognormal distribution would create a lower limit to a random number generation and will provide a positive skew, which is very important for simulation efforts. Using optimized return and standard deviation obtained from MVO equations, the expected value and standard deviation of the natural log of the return relative of the portfolio are calculated as:

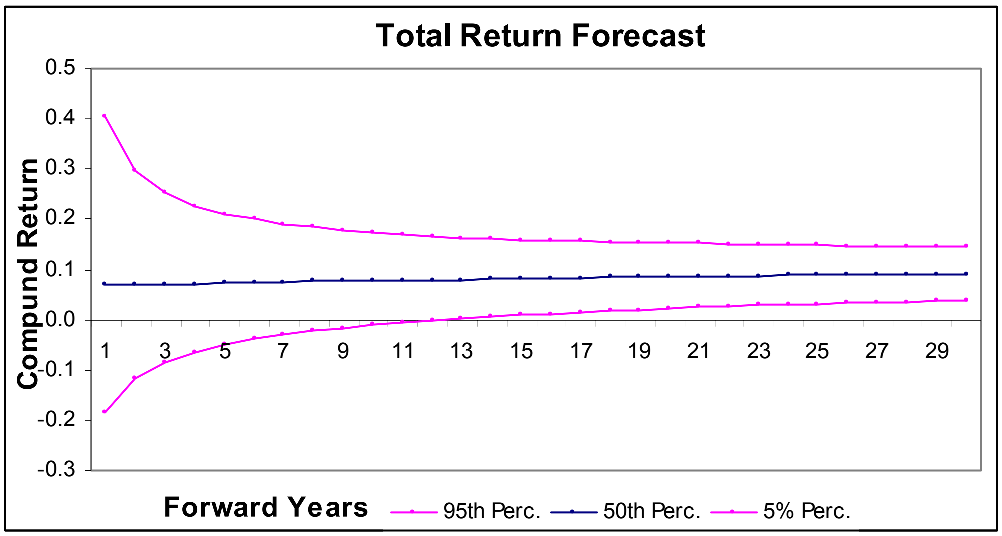
$$m = \ln(1 + u) - (s^2 / 2)$$
$$s = \sqrt{\ln\left[1 + (\sigma / (1+u))^2\right]}$$

The probabilities of cumulative wealth, wn, and the probabilities of various compound rates of return across time, r_g, can be calculated by:

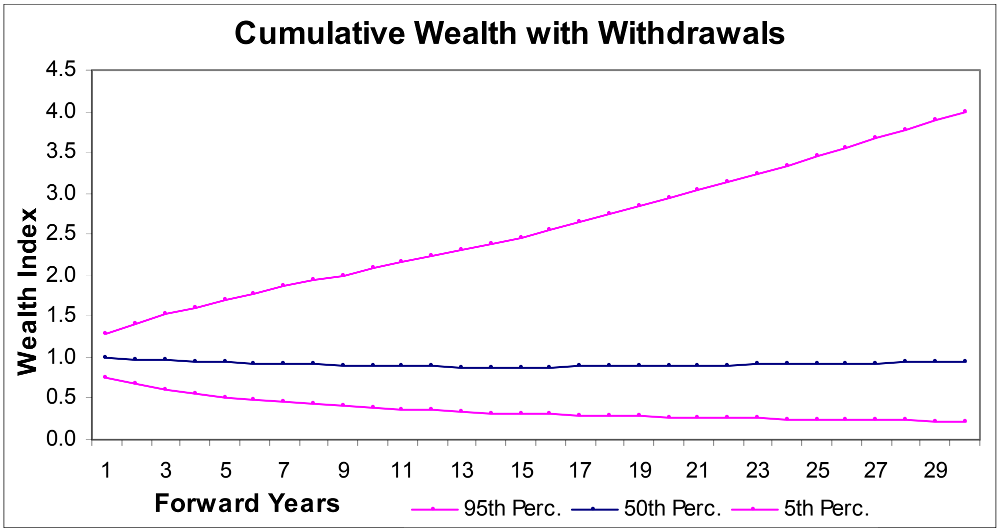
$$W_n \text{ percentile} = e^{(m\,n + z\,s\,\sqrt{n})}$$
$$r_g \text{ percentile} = e^{[m + z\,(s / \sqrt{n})]} - 1$$

High and low percentiles of the compound rate return (95^{th} and 5^{th} percentiles, for instance), for all periods of 1 to n, will converge in n periods upon the median 50^{th} percentile, resulting in a trumpet graph. A tulip graph will be generated from the percentiles of cumulative wealth on n periods, showing the ever-widening range of wealth over successively longer time periods. The following graphs show probabilities of cumulative wealth and annualized compounded rates of return. A DRRA assumption is impliedly contained in the assumptions, as optimization gradually increases return rates over the years, due to a shifting into equities. An IID assumption is also implied in the graphs, as the standard deviation remained at levels associated with a one-year frontier.





The above graphs are consistent with an investor who is the accumulation phase of his or her life, as only assets have been modeled so far in the model. This is a deferred compensation pattern that typically occurs during the work years of one's life. Liabilities can be easily put into the equations as an annual percentage withdrawal from cumulative wealth. The withdrawals can be adjusted for inflation as well, to reflect the loss of purchasing power over time. The withdrawal can be modeled simply by subtracting the withdrawal rate from the expected rate of return of the assets. In the following graphs, the same assumptions are used as initially, but the withdrawal rate is now assumed to be 5%, and the inflation rate is assumed to be 3%. Also, the implied DRRA assumption through optimization is now changed to a CRRA assumption with a fixed 25% bond percentage in all time periods, since it is unlikely that an investor would be withdrawing assets from a portfolio yet simultaneously increasing equity percentages with increases in wealth. Thus, the following graph is more typical of an investor who is retired, with CRRA and IID investor-level preferences, and who is now starting to consume investments through gradual withdrawals.



Note the dramatic difference in results. The median cumulative wealth projection decreases slightly into the long-time frames and only marginally increases very late in the projection. There is also around a 50% probability that annualized returns will actually be negative in the very long holding periods. This is due to the withdrawal rate of 5% per year and 3% annual inflation taking away most to all of the gains made to wealth from investment returns. With return rates of the portfolio ranging from 8% in near-term time frames to 9.65% in the longer holding periods, it is not difficult to understand why there may be losses to net worth once withdrawal commences.

The above forecasts assume that the sequence of cash withdrawals don't make a difference in net worth calculations. In reality, the cash flow sequence matters greatly. This is because actual returns may have several consecutive years of highly negative or highly positive results before returning to a long-term trend line. Recent historical events demonstrate that rolling years of very positive results can be followed by several years of highly negative results before returning to the long-term trend (e.g. very good returns in the mid to late 1990's, followed by highly negative returns from 2000 through 2003). While the use of probability curves enables an analyst to project a range of probable ending wealth, reliance upon expected compound rate of return makes the estimations dependent upon median percentiles and incapable of modeling a cyclical type of return scenario. End period wealth merely becomes a function of the average rate of return and the range of returns. This would not pose much of a problem if cash flows were not being regularly withdrawn from cumulative wealth. In a 2000 to 2003 type of occurrence however, before the returns could come back to expected levels, a portfolio's cash reserves could be quickly depleted by continual annual withdrawals. Thus, the economic projections should be considered as producing only average or expected estimations.

Monte Carlo Simulations [22]

To make the forecasting model more realistic, simulations are needed. The probability of a certain event being modeled (i.e., the shortfall of investment objectives) can be estimated by running repeated simulations of an economic model. This greatly improves the quality and usefulness of results, for now the possibility of cyclical patterns in portfolio returns can be modeled. Indeed, Monte Carlo is noted for random number generation, so that all types of return patterns can be observed over thousands of computer runs of the model. Monte Carlo allows analysis of complex financial systems by generating approximate solutions through random number generations, given various probability distributions.

Various types of simulations exist. Models can estimate individual asset returns, although the confidence level of the results will be lower than with asset class simulations, due to the greater volatility of individual assets. Historical models (sometimes referred to as back-tested models) use only historical data for assumptions and inputs. Parametric simulations are based on estimations of the inputs. Random samples are then generated from a probability distribution, thereby going beyond merely historical data. Macro-economic models will estimate changes in forward economic variables that can impact risk premiums and rates of return. Thus, forward returns and other inputs can vary in accordance with projected changes in the macro-environment. Monte Carlo simulations are usually considered to be parametric in nature, allowing the researcher to vary input assumptions beyond a historical data set.

The following outline contains the important procedures for Monte Carlo simulations.[23] The outline has been adapted to fit the following example of a $1 Million portfolio with cash flow withdrawals.

1. Specify the starting values of the underlying variables of the asset and liability stream.

2. Specify the time horizon and decide on the appropriate intervals.

3. Determine the types of distributions for the risk factors driving the underlying variables.

4. Draw random values for each risk factor of the simulation. .

5. Use the random values to calculate the underlying variables.

6. Calculate quantities of interest and arrive at final end-period wealth. This is one simulation trial.

7. Run the simulation for x number of trials.

8. Produce descriptive statistics of the simulation. VaR-like conclusions may also be able to be made.

Monte Carlo simulations can be estimated by:

$$w_t = (1+r_t)\,(w_{t-1} - aw_{t-1} + (aw_{t-1} * p))$$

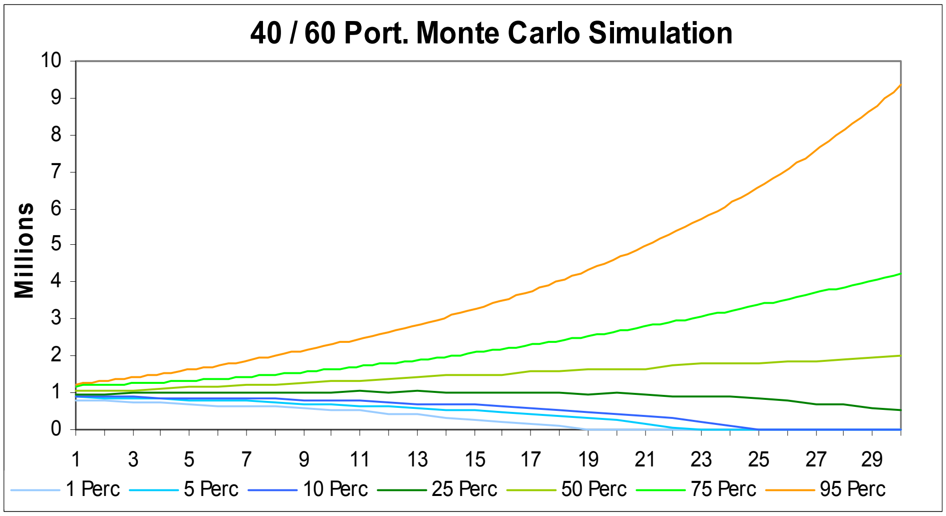
where, w_t is wealth at year-end t; r_t is the total return in period t; aw is the annual withdrawal, and p is the inflation rate or COLA used (if any). r_t is randomly generated within a lognormal probability distribution whose mean and standard deviation is determined by the assumptions to the model.

With many simulations, a lognormal distribution is used because a normal distribution could overestimate extreme values with a potential for infinite gains and losses. Even where the returns of a diversified portfolio are not lognormally distributed but are independent and identically distributed (as assumed in this simulation), cumulative end period wealth and the rates of return are approximately lognormal in distribution.[24]

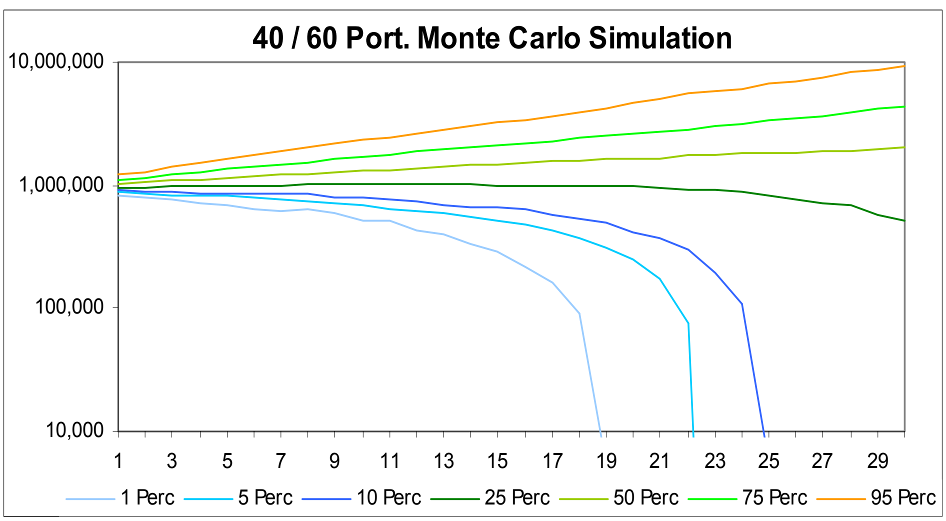
The need for regular cash withdrawals at retirement often will produce fixed equity and bond allocations, with rates of return and standard deviation typical of short-term holding periods. The investor may be long-term in his or her forward investment perspective, but the scope and timing of withdrawal patterns may obviate mean reversionary abilities.[25] In fact, where there is a constant requirement of cash withdrawals from an investment portfolio, the investor would always be facing a one-year (or less) time horizon, complete with very high levels of pricing volatility.

The following graphs use the same probabilistic concepts that were discussed in the Time chapter. Starting with an assumed $1,000,000 portfolio, historical data of the broad-based asset classes are optimized on a one-year historical frontier. The simulation also assumes that the investment objective is to have cash withdrawals of $50,000 per year plus an annual inflation adjustment of 3% per year. The objective calls for 30-year horizon. The portfolio is assumed to contain a fixed allocation of 40% equities and 60% bonds. No taxes, trading expenses, or costs of administration are assumed.

The simulation was run 500 times with random numbers generated for the portfolio rates of return in each year, assuming mean and standard deviation of the optimized portfolio and further assuming a lognormal distribution of returns. [26] Notice the remarkable spread of portfolio values.



The probability of shortfall can be more easily observed when the same data is placed on a logarithmic scale, as is done in the following graph. The probability of shortfall is now displayed as confidence levels. Portfolio failure occurs at the 1 percent level of confidence by year 19; the 5th percentile by year 22, and the 10th percentile by year 25. Even the 25th percentile is in decline by year 30, and will ultimately fail, given enough time. For the assumed asset and liability streams, portfolio failure over indefinitely long horizons may be as high as the 35th percentile.



<u>In summary of Monte Carlo simulations</u>, the probability of not fully funding investment goals can be capably forecasted. The method is easier to develop than Asset-Liability Modeling since it avoids many accounting and actuarial concerns associated with ALM. In this regard, economic forecasting efforts are perfectly suited for evaluations of investor-level shortfall as well as macro-level forecasts. Multi-individual plans will likely

still prefer ALM modeling however, due to the interplay with federal regulatory issues and complexity of cash flow management. Economic models, and in particular, Monte Carlo simulations, are also very sensitive to various assumptions and inputs. Results will also vary somewhat between simulations using the exact same data, equations, and assumptions, due to the random nature of the variables being simulated. This problem can be partly overcome by increasing the number of simulated runs to produce a better convergence of results. But conclusions reached by simulations should be generally considered fluid in nature and only indicative of estimated probabilities.

Value at Risk [27]

This method was initially developed by financial firms in response to a series of financial disasters in the derivative arena in the 1980's and early 1990's. Asset-Liability Modeling had been in common usage, but difficulties with actuarial methods (some of which were noted in the above section on ALM) led to a search for a replacement. Till Guldimann of J.P. Morgan attempted to keep market value of a portfolio constant by analyzing variance over short time frames. The purpose initially was to measure the risk exposure of a firm's trading positions. The Chairman of J.P. Morgan at the time, Dennis Weatherstone, implemented VaR throughout his firm and obtained a daily VaR report of the total market risk exposure of the business. The first widely circulated paper on VaR came in 1993 in a Group of Thirty Report. J.P. Morgan then disseminated a "Risk Metric" system in October, 1994. In 1997, the SEC required a disclosure of quantitative information on derivatives and other financial investments. The Basil Bank Accords of 1988 were amended in 1996 to include market level risk, and again in 1999 to include credit risk. Both of these revisions used VaR-like concepts to refine notions of risk management used in the banking industry. From these beginnings, VaR has become a leading tool to study firm-wide risk management.

A Basic Discussion of VaR. Financial risk is typically seen as being composed of market risk, credit risk, liquidity risk, operational risk, and legal types of uncertainty. While the original purpose of VaR was to quantify market level risks, the analysis has evolved to risk management across an entire firm. VaR analyzes possible losses over given time horizons, at specified levels of confidence. While the focus is on the probability of losses in normal market environments, the issue can be extended to extreme or worst case scenarios through stress testing. The basic question is framed in this manner: Over a given time horizon and for normal market conditions, what is the worst loss that can happen at a specified level of confidence? The expected distribution of losses can be expressed as a single number.

VaR is usually developed relative to the expected value of a variable such as portfolio value or earnings per share, depending upon the inquiry. Normal VaR uses a normal distribution to estimate the probability of the expected portfolio value over short time horizons. The basic equations are:

$$\text{VaR} = \text{portfolio } \$ * \sigma * \sqrt{(\text{t-days} / \text{annual days})} * z$$

Normal VaR = - [(Exp. Δ value over t-day horizon) – 1.645 * (σ *Δ value over t-day horizon)]

For example, assume that the expected change of a portfolio valuation is $10 Million over a seven-day horizon and that σ to the portfolio value over the same time period is $18 Million. At the 95% confidence level, a $20 Million loss may be equaled or exceeded 5% of the time.[28]

VaR analysis goes farther than standard deviation types of risk with a probability boundary of potential losses being established through VaR. Standard deviation is seen as a single point estimate within a broader probability distribution. With VaR, portfolio risk is thus seen in terms of the entire probability statement. VaR represents an extension of valuation methods for derivatives first pioneered in the Black-Scholes formula. Stress tests can then be used to evaluate the worst possible loss (i.e. the tails of the probability distribution).

The volatility of returns as well as the exposure to the volatility are both measured by VaR. One-month time horizons are typically assumed, although banks will often calculate a daily VaR because of the time required to hedge market risks, as well as VaR being needed as a statistic on daily profit and loss statements. As the time horizon lengthens, the VaR estimates will become closer to the expected value. VaR will generate a trumpet graph of probability that is very similar to the time-referenced portfolio return graph and the cumulative wealth graph noted in the last several chapters. Lack of serial correlation across time horizons, which is consistent with efficient market theories, is also usually assumed. Normality of probability distribution is also assumed, as is the lack of changes in portfolio composition over the assumed time horizon.

VaR assumptions are ideal for the fast paced environment of financial instruments and derivatives. VaR loses much of its usefulness where time horizons are stated in years instead of days or where portfolio returns are not normally distributed. Additionally, where fundamental notions of business risks are being measured, such as the volatility of cash flows, the lack of serial correlation becomes a troubling assumption. Poor cash flows in a business are normally followed by poor cash flows in the next period. Long-term asset return predictability also poses problems for extending VaR beyond its typical assumptions.

The steps to calculate VaR include:[29]

1. Mark positions to market (value);
2. Measure variability of risk factors (σ);
3. Set time horizons (t-days);
4. Report potential losses in standard units.

Following these steps, an example of VaR is:

1. A portfolio is marked to a market value of $100 Million;

2. This is multiplied by standard deviation of 15% (or .15);
3. For a given √ t-days of 10 days / trading days in year (or √ 10 / 252);
4. At a given confidence level, say 99% (* 2.33);
5. The result is Value at Risk of $7 Million.

The process is greatly simplified if the probability distribution is parametric in nature (such as is the case with a normal distribution). VaR can then be calculated directly from portfolio standard deviation. A normal distribution increases the power of the estimate in that the emphasis will be on the spread of the distribution, and not just a single point estimate.

Enhancements to VaR. In additional to Normal VaR, other types of calculations are possible. The analysis can be enhanced with Marginal VaR. The change in portfolio VaR can be reviewed by looking at the added dollar value of exposure for an additional dollar value to a component part of the portfolio. Systematic risk of a portfolio establishes the base-line measure, and then the marginal difference in risk levels can be measured as a result of additional dollar values of assets. Incremental VaR is similar, making a before and after comparison. Component VaR measures a complete deletion or addition of a component of the portfolio. These models can be further modified by using generalized autoregressive equations (GARCH). Long horizon forecasts are then possible, and variance can also be modeled using an exponentially weighted moving average (EMA). An implied standard deviation can be calculated by looking at the market price of an option for market value. The implied volatility can be determined through an estimation of future volatility of the option. Such an option based forecast may produce better results than relying only upon historical measures of volatility. VaR can be extended into simulation analysis, complete with probability boundaries and confidence levels. It can also encompass stress testing to look at extreme scenarios.

While market-level volatility risk can easily be incorporated into VaR analysis, other forms of risk are more difficult to quantify. Credit risk involves the probability of loss due to a counterpart's failure to perform an obligation. This form of risk could be modeled as a short option, or as a potential loss of default in forward time frames. To account for impacts to the portfolio from credit risk over time, Monet Carlo Simulations could be run. An attempt at evaluating liquidity risk can also be made, but VaR normally relies upon market pricing to do so. For non-financial corporations, the focus is on cash flow, and the probability of certainty of the cash flow. The process can be generalized however, to all earnings and not just cash flows of a business.

Risk adjusted performance measures (RAPM) can be developed. This is basically a risk adjusted return to capital, known as RAROC. RAPM is usually based on non-diversified or individual VaR. Component RAPM measures the diversified levels of VaR.

Shareholder value analysis can be conducted. Some financial institutions use VaR to select projects or businesses for expansion only when RAROC's are high. VaR is also consistent with EVA and value added concepts, since EVA is simply the numerator for RAROC. Some banks consider projects to add value when RAROC > cost of bank's equity. The only major difference with value added models is that VaR uses multi-period analysis whereas EVA is a one period calculation.

In summary of VaR, the method has much to offer in the evaluation of downside risk. The probabilistic nature of VaR adds great sophistication to risk analysis. But the traditional notions of VaR must be supplemented by stress testing to analyze extreme possibilities. So far, VaR has been primarily used to evaluate financial risk in very short time frames. It has encountered difficulties in analyzing fundamental risks of a business as well as the non-market investor-level forms of risk. For fundamental notions of business risk, cash flow default models (e.g. Altman Z scores & bankruptcy probability models) may be better suited to the task. Indeed, Monte Carlo simulations are thought to work better than VaR at estimating cumulative changes in firm value and default probabilities over extended time periods.[30] For shortfall risk involving complex contribution and vesting streams, Asset-Liability Modeling may be more appropriate, as discussed in above sections of this chapter. Even where VaR can apply, it should be viewed as providing only an estimate of risk management at a firm level. Due to the probabilistic nature of VaR, exact answers and definitive answers are not generally possible. Instead, the conclusion is reached to a certain degree of confidence.

Conclusion

Considerations of shortfall risk go back to the earliest beginnings of portfolio theory, starting with A.D. Roy in 1952. Since shortfall risk is felt at the investor-level, market-level forms of risk and return are primarily relevant in determining the probabilities of the forward asset stream. The other half of the equation, that of the investor liability stream, must be matched with the asset stream to produce a high degree of confidence that investment objectives will be achieved. Asset-Liability Modeling is a wonderful tool to evaluate forward net cash flow streams, especially for multi-individual portfolios. Economic forecasting models are ideal in many other situations, including individual and firm investment portfolios. Value at Risk is largely suited for financial risk management, and is currently being expanded to include firm-wide risk management.

Some common themes emerge from the various models reviewed in this chapter.

1. Risk and return should be viewed in probabilistic terms. The expected rate of return and standard deviation of return should be treated as single point estimates of a broader probability distribution of return. Risk should entertain the entire probability density function, including skew and kurtosis, and the downward shifts in the probability distribution.

2. While the asset side of investments focus on market-level pricing factors, the liability side occurs at the investor level. Non-market, individual risk factors and preferences therefore become crucial, and such factors should be considered for each investor. Once liabilities are brought into the discussion, risk should be seen as the probability of failing to meet investment objectives, for the given time horizon and give utility.

3. By using various aspects of MVO, VaR, utility, accounting models, and economic simulations, a more sophisticated and heightened understanding of portfolio management than what is possible with just an asset-only trade-off of pricing risk and return. Extending MVO to include liability models is still consistent with the special case of utility maximization. Market level information and investor level interests can both be capably analyzed within the constructs of a multi-period asset-liability model.

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[1] Much of this section is based on DeFusco, et al (2001), at 253-254, with historical anecdotes taken from *Capital Ideas: The Improbable Origins of Wall Street,* Peter L. Bernstein, The Free Press, 1992.
[2] Roy's paper was "Safety-First and the Holding of Assets," A.D. Roy, *Econometrica,* 20 (1952): 431-439.
[3] Martin L. Leibowitz and Roy D. Henriksson, "Portfolio Optimization with Shortfall Constraints: A Confidence-Limit Approach to Managing Downside Risk", *Financial Analyst Journal,* Vol. 45, No. 2, 34-41 (1989).
[4] The equations and procedures are noted in Elton, Gruber, et al (2003) at 235-241; 250-254.
[5] This is also from Elton, Gruber, et al.
[6] William F. Sharpe, "Mutual Fund Performance," *Journal of Business* 39, no. 1, part 2 (Jan. 1966): 119-138.
[7] For a small discussion of BTR and BTP, with citations to more extensive research on the topic, *see*, *The Revolution in Corporate Finance,* Joel M. Stern and Donald H. Chew, Jr., editors, (4th ed. 2003), Chapter by Culp, Miller, and Neves, at 426.
[8] This section is taken from reading materials pertinent to CFA III study guides, risk management study sessions (circa, 2005).
[9] See, "On the Risks of Stocks in the Long Run," Zvi Bodie, *Financial Analyst Journal* (May / June, 1995:18-22).
[10] "An Analysis of Investment Advice to Retirement Plan Participants," Zvi Bodie, Pension Research Council Working Paper, 2002-15.
[11] But this argument also rests on the CRRA and IID assumptions. When the assumptions are relaxed, dramatically different return probabilities can arise.
[12] "Beware of Dogma," Mark Kritzman and Don Rich, *Journal of Portfolio Management,* Summer, 1998, p. 66-76.
[13]This section summarizes several very good presentations made at the 2005 Annual Conference of the IFEBP. See, the Bibliography for specific presentations.
[14] The procedural steps are generally taken from "Asset Liability Modeling", Allan R. Ettinger, CIMA, presentation made at the IFEBP Annual Conference, 2005.
[15] The PPA of 2006 changed multi-employer plans of assumption reasonableness in the aggregate to the single-employer plan rule of each assumption, individually, being reasonable. See, "Trends in Actuarial Assumptions," Cary Franklin, FSA, and Ira M. Summer, FSA, 2006 IFEBP Annual Conference.
[16] See, SEC June 1993 letter to regulated corporations.
[17] "Assessing Your Plan's Asset Liabilities," Mark T. Ruloff, 2006 IFEBP Annual Conference, at TMPG3-49.
[18] The equations are contained in "The Funding Status of a Multiemployer Plan," Bruce Cable and Mike Rust, IFEBP Annual Conference, 2005; and "Assessing Your Plan's Asset Liabilities," Mark T. Ruloff, 2006 IFEBP Annual Conference.
[19] The immunization concept was introduced by a British actuary in 1952, F.M. Redington, "Review of Life-Office Valuations," *Journal of the Institute of Actuaries; Much of this section is a summary of:*

"Portfolio Immunization", Allan R. Ettinger, CIMA, International Foundation of Employee Benefit Plans, Annual Conference (2005).

[20] Portfolio standard deviation could actually go down since the portfolio would be more heavily weighted towards short-term investments having lower volatility.

[21] This section summarizes many cumulative wealth concepts contained in Forecasting and Optimization sections of the *SBBI Yearbook*, Ibbotson Associates, (2005).

[22] This section on Monte Carlo draws heavily from both Ibbotson (2005) and DeFusco (2001).

[23] The procedures are generally taken from DeFusco, et al (2001), at 262.

[24] See, Ibbotson (2005) for details on using lognormal distributions in Monte Carlo simulations.

[25] This was demonstrated by Paul Samuelson in "The Judgment of Economic Science on Rational Portfolio Management: Timing and Long-Horizon Effects", *Journal of Portfolio Management*, 16 (1989), 4-12.

[26] The assumptions and procedures closely follow and emulate a simulation developed by Ibbotson, *SBBI Yearbook* (2005), Ch. 10.

[27] Many of the concepts in this section are taken from Jorion (2001); and Stern & Chow, ed. (2003), chapter by Culp, Miller, and Neves.

[28] The equations and example are from DeFusco, at 255-257.

[29] The steps and example following are taken from Jorion (2001), at 113.

[30] Stern and Chew (2003), at 380; 425.