General Utility Concepts Applicable to Investment Choices

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Forward

Earlier working papers (circa 2005 and 2006) focused on extending the special case of utility maximization to include time horizons. These efforts culminated in a paper outlining a "time horizon extension" to Modern Portfolio Theory (2006:Summary). That summary became the basis for an article published in the IFEBP Benefit and Compensation Digest in early 2007 (Kaufhold, 2007). The working papers also served as the basis for several chapters in a draft book on the long-term investing principles (Kaufhold, draft).

As was suggested in the closing paragraphs of 2006:Summary, the general case of utility and dynamic models may be necessary to fully integrate time horizons into portfolio theory. Extending the special case of utility to include time horizons results in useful and workable techniques for both theoreticians and practitioners. However, the quadratic equation used with means-variance procedures inappropriately models absolute risk aversion, and thus makes any MVO-styled investment process ill-suited for explaining investor behavior across varying holding periods. The general case of utility maximization provides a more thorough and accurate explanation of how investor behavior can be modeled over successively longer time periods.

This paper should therefore be considered as a general utility companion to prior investment modeling efforts of the author that used the special case of utility. This paper does not break any new theoretical ground, however. It merely attempts to collect basic information and equations regarding the dynamic modeling of investments across time. This working paper is essentially a summary / outline of several books and papers on utility. Unless otherwise noted, the paper follows and outlines Eeckhoudt, Gollier, and Schlesinger (2005). Special reference is also given to Gollier (2001); a utility chapter in Elton, Gruber, et al (2003); and several papers, where noted. See these texts for a more thorough analysis of the subject matter.

The Historical Development of Utility Theory

As early as the 17th Century, Pascal, Fermat, and other scholars believed that the value of a lottery should be equal to its mathematical expectations, and therefore identical for all people, independent of their risk attitude. We see that same notion surface today in the market portfolio concept, whereby all investors should hold the optimal portfolio, regardless of risk tolerances and individualized considerations.

Daniel Bernoulli, penned a paper in Latin in 1738, while at St. Petersburg. Bernoulli introduced the famous St. Petersburg paradox in this paper, as well as providing two other examples to illustrate his thesis. Bernoulli's conclusion was radical for the time: the mathematical expectation of the monetary outcomes is not an adequate measure of its value. Instead, a lottery should be valued by the expected utility that it provides.

Bernoulli was the first person to suggest a nonlinear relationship between wealth and the utility of consuming that wealth. The real question becomes how much happiness is to be derived from the monetary outcome, rather than possessing the monetary outcome itself. The relationship between satisfaction and the monetary outcome, x, can be described as a utility function, u (x), which is attained by the agent as a result of possessing the wealth. The monetary outcome is objective, but the satisfaction is subjective in nature, depending upon the preferences of each agent. u(x) becomes an indirect measure of wealth. u(x) is the highest attainable level of utility for a given set of goods affordable with the individual's income.

Bernoulli believed that if utility is increasing and concave in outcome x, then a person or agent (as a person is referred to today as) would prefer the concave function over other types of outcomes. The concave function could be $u(x) = 2*\sqrt{(x)}$, or $u(x) = 3*\sqrt{(x)}$, for instance, and a linear function, $u(x) = x$.

A concave function suggests that marginal utility of wealth is decreasing with an increase in wealth. One of Bernoulli's example involved diversification of the risks of loss. The mathematical expectation of the outcome of a lottery is the same, with or without the diversification. Bernoulli used the example of ships carrying freight, with one in ten of the ships perishing in the voyage. Do you carry all the freight on one ship, or more than one ship? Mathematically, the outcomes are the same either way. But most people would prefer to carry freight on more than one ship. Their satisfaction, or utility, increases by diversifying against the risk of loss. A concave utility function with declining marginal utility would generate more utility with diversification than would a non-diversified risk.

Francis Edgeworth introduced the generalized utility function U (x, y, z, ...) into the economics profession and drew the first indifference curve in 1881. Edgeworth's original two axis depiction was then developed into the now familiar box diagram by Vilfredo Pareto in his *Manual of Political Economy* (1906). Pareto effectively began the field of modern microeconomics when he developed the notion of Pareto-optimality. The society enjoys maximum optimality when no one can be made better off without making someone else worse off. Arthur Lyon Bowley popularized the Edgeworth box in *The Mathematical Groundwork of Economics* (1924), so much so that it is commonly known as the Edgeworth-Bowley box.

Two centuries later, von Neuman and Morgenstern (1948) refined Bernoulli's thoughts into the modern-day expected utility (EU) model. This amounted to a major breakthrough, as theoretical depiction of economic choices could now be expanded to include both conditions of certainty and the probabilities of an event was uncertain (i.e. risky). The model relies upon the following axioms. See, Elton, Gruber, et al (2003).

Comparability – the investors are assumed to make comparisons between certain outcomes.

Transitivity – investors are consistent in ranking their outcomes. If A is preferred to B and B to C, then A is preferred to C. In complex situations, it may be difficult to clearly ascertain a ranking of outcomes.

Independence – An investor is indifferent between two equal probabilities.

Certainty Equivalent – for every gamble, there is a value at which the investor is indifferent between a gamble and a certain equivalent.

These assumptions are important for use with indifference curves, which have become a standard way of describing economic choices under uncertainty. As shall be shown in a few pages, based on these assumptions, an investor's gamble can be mathematically expressed as the probability of the utility of wealth. This is essentially a simple mathematical definition of expected utility.

Shortly after WWII, a mathematician, Richard Bellman, pioneered dynamic programming efforts. By 1953, Bellman had developed a mathematical model that studied the optimization of variable as they approached infinity. Today, Bellman's works are used in such diverse areas as operations research, computer programming, theoretical particle physics, and economics. This type of modeling is playing a critical role in the further development of the multi-period investment and consumption models.

Harry Markowitz heavily relied upon Expected Utility in designing the efficient frontier (1952, 1959). Markowitz showed that utility was maximized along the frontier. Thus, standard deviation of pricing returns could be used as a measure of risk instead of having to calculate utility. It was sufficient to know that utility was maximized at an optimal level of standard deviation occurring at all points along the efficient frontier. With the development of the CAPM in 1964-1966, the optimal level of pricing risk and return became identified as one point on the frontier, at the tangency between the CML and the frontier.

Then, Pratt (1964) and Arrow (1963) independently developed absolute and relative risk aversion concepts. With this development, Expect Utility matured into a model that could explain economic choices over a wide variety of factors and assumptions. Samuelson, Merton, and Mossin all led efforts in the 1960's and 1970's exploring inter-temporal modeling of assets using utility concepts. More recently, Gollier and many others are extensively researching and writing on lifetime consumption models.

Pros and Cons of General Utility

The general case of utility and dynamic programming efforts provides significant advantages for the inter-temporal modeling of assets and liabilities. It allows for the modeling of any risks, thus obviating the necessity of the normality assumption under MVO. Any form of absolute and relative risk aversion can be modeled. This is unlike the quadratic utility function, which inherently is increasing in absolute risk aversion. Return predictability and many other macro and investor-level factors can also be expressly modeled. Most importantly, asset allocations can be varied in short-term time frames in order to maximize the long-term accumulation of wealth.

Disadvantages of the dynamic process exist, however. The concepts and equations are highly theoretical and complex in nature. They are not overly amenable to immediate, practical applications. While dynamic programming is well-known among utility theoreticians, so many investor-level factors have been identified in the literature that it may be virtually impossible to unequivocally state the appropriate allocations and investment choices facing agents across time. Additionally, dynamic investment equations are essentially part of general equilibrium consumption models. These models are even more theoretically removed from the daily world of investments, so much so that it is common to speak in terms of representative agents instead of the more practical investment choices of individual investors.

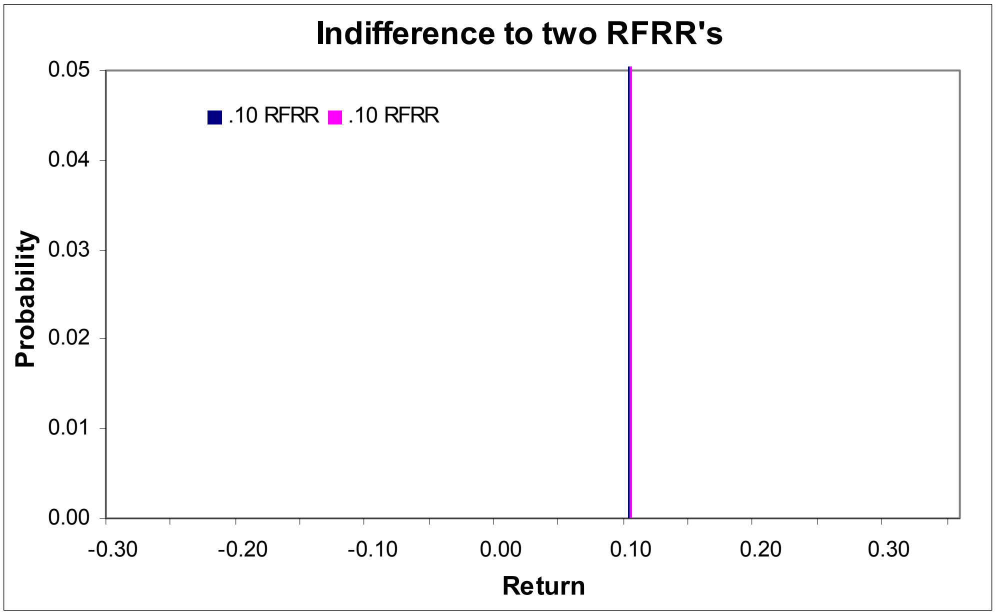
Still, dynamic portfolio management theoretically explains the wide variety of economic choices facing investors. It considerably expands the analysis, and provides additional tools and concepts that can be used in the maximization of capital wealth across an investor's entire life.

The theoretical modeling of investments of the special and general cases of utility should therefore be viewed as a progression of investment-related concepts. MVO operates within an exclusive investment universe of pricing volatility, and is basically an asset-based optimization system using pricing variables with short-term holding periods (typically one-year in duration or less). To some extent, the time horizon extension proposed in 2006:Summary, as well as 2007:3, translates MVO into an asset-liability model applicable across varying time horizons. The relevant risk changes from pricing variance to the probability of shortfall from meeting investment objectives. The dynamic management of wealth then contemplates a consumption-styled universe where investment decisions are considered to be deferred consumption. The relevant risk becomes the variability of lifetime consumption. Investment choices under conditions of uncertainty and across time can be fully modeled with dynamic methods.

The Nature of Risk

Risk and return choices and trade-offs have been noted in several previous working papers. The following discussion provides graphical representations along with a few illustrative equations.

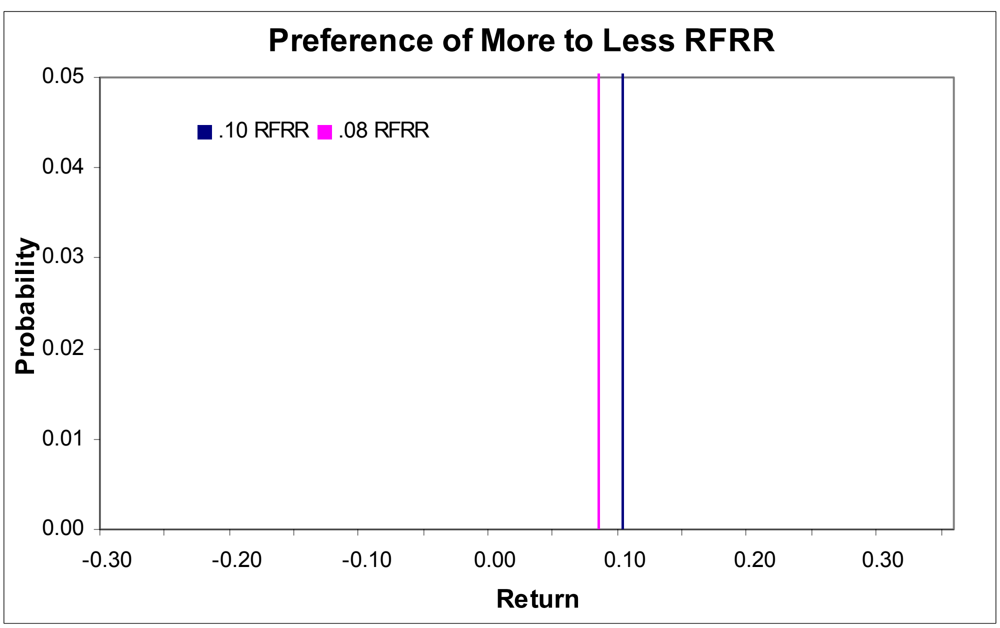
We start off with an investor having a choice of two risk free assets. The individual will be considered indifferent to risk free assets having the same rate of return. With both assets, there is a 100% probability of achieving the identical return, assumed in the example to be 10%, or .10.



For purposes of convenience, only the first 5% of probability is displayed. By assumption, there would be a 100% certainty of a 10% return outcome. The utility of each return would be the same. With wealth state w_1 and w_2, this is mathematically described from the following equations. The formulas and general analysis of stochastic dominance are taken from Eeckhoudt, et al (2005).

$$u(w_2) = u(w_1)$$

Let's now give the individual a choice between two risk-free assets, one with a 10% return and the other with an 8% rate of return.

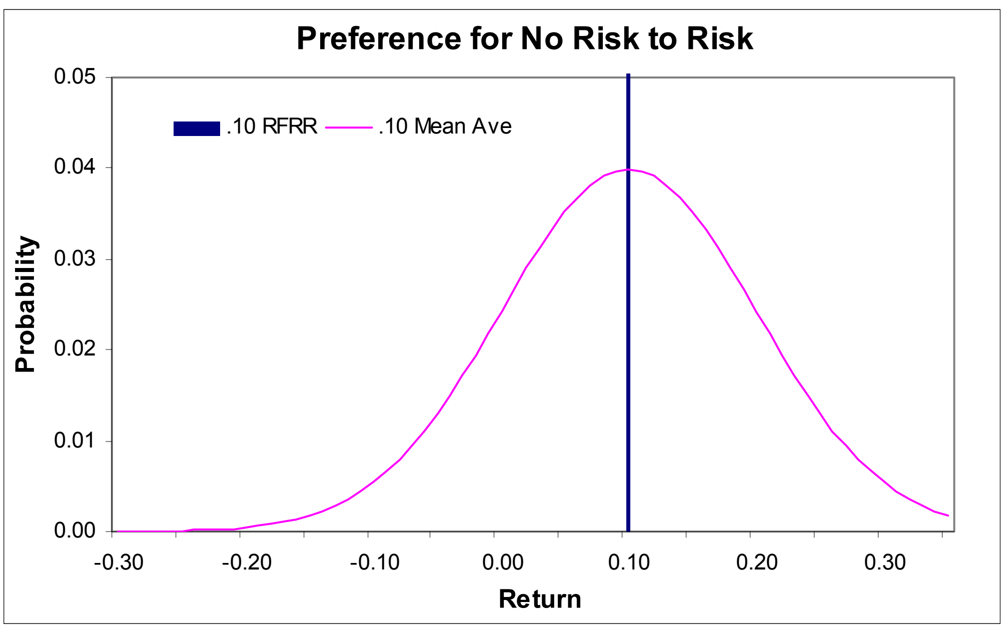


The investor will clearly prefer the choice of a greater return to a lesser return, so long as both assets are completely certain of generating the quoted rate. This served as the origin for the rational investor assumption of modern portfolio theory, as all investors are generally assumed to want more to less. Where w_2 is now the wealth state with the lower rate of return, the situation is now:

$$u(w_2) < u(w_1)$$

In reality, most investment choices are not so clear. The risk-return trade-off is often present, with a higher risk having to be taken for a corresponding higher return.

All risk-averse individuals will prefer less risk to more risk. The following graph depicts the investment choice between a risk free rate of return with a 100% probability of 10% versus a return having the same expected mean average of 10%, but now having some variability of that expectation. The variability of return is considered as the relevant risk in the example, with a standard deviation of 10%, or .10, a skew of 0, and excess kurtosis of 0. As students of statistics will know, the variability of these three precise moments of the distribution will sketch out a normal probability function.

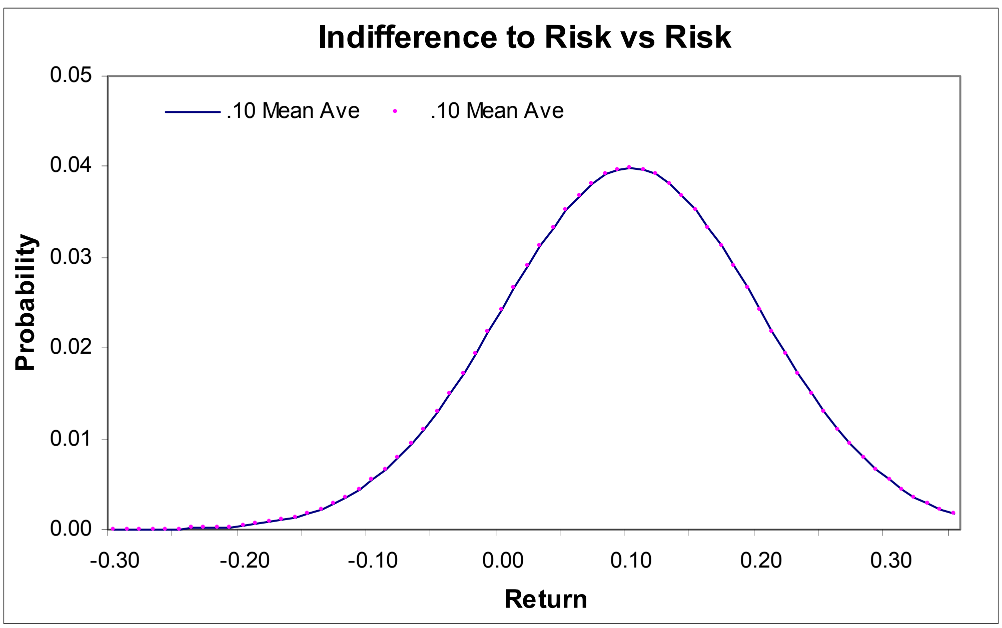


All risk-averse agents will now prefer the risk free asset versus the risky asset having the same expected rate of return. Why take risk without any corresponding increase in return? This is the basis for the risk-averse investor assumption in portfolio analysis. With probabilities of return now being inserted into the equations, the above graph can be mathematically stated as:

$$Eu\ (w_2 + z\sim) <= u\ (w_1)$$

where, z~ is a zero-mean risk, and Eu is expected utility, based on the probabilities of return. A zero-mean risk occurs where the expected pay-off of the risk is zero, or Ez~ = 0. The mean average return has not changed, but there is now risk from the existence of a range of possible returns across an entire probability distribution. The utility of certain wealth will be preferred to the expected utility of wealth having risk attached to it. In the case of variance of return, this mathematical representation is correct only if we additionally assume normality and quadratic utility. In the more general case, the addition of any risk, whether normal or not, could introduce a zero-mean risk into the analysis.

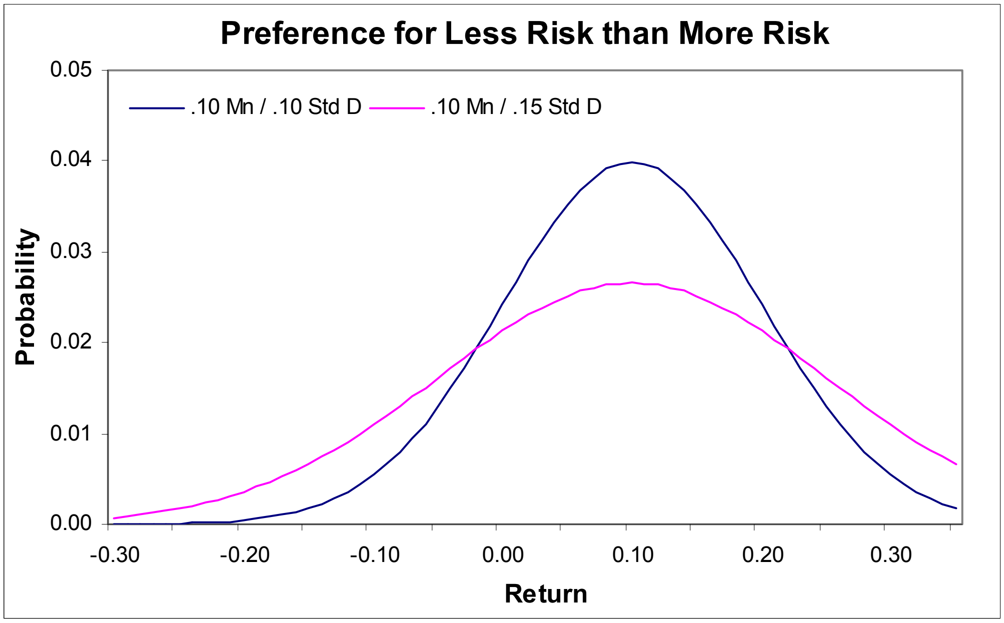
Now, let's assume that the investor's choices are between two risky assets, both of which have the same, identical rate of return and variability of return. In the following example, both risk assets have 10% expected return, 10% standard deviation, a skew of 0 and an excess kurtosis of 0.



The agent will once again be indifferent to the two investment choices, since they both offer the exact same rate of return, and with the exact same variability of return. The expected utility of both distributions will be the same, again assuming the special case of normality and quadratic utility:

$$Eu\ (w_2 + z\sim) = Eu\ (w_1 + z\sim)$$

Just as there is a general preference to take more return for the same level of risk, so too will agents have a preference for taking less risk for the same level of return. This is shown in the following graph.

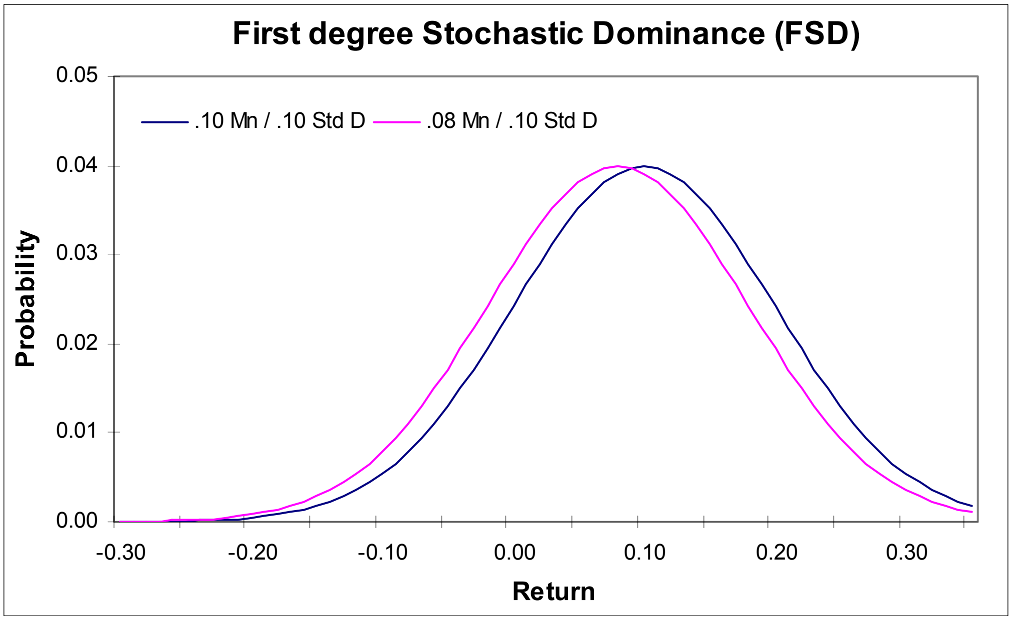


In the above example, the two choices have the same 10% level of return, but the second choice now has a greater variability of return. The blue distribution is a normal curve of 10% return, 10% standard deviation, 0 skew and 0 excess kurtosis. The pink distribution has 10% return, 15% standard deviation, 0 skew and 0 excess kurtosis. Both are normal curves, but the second choice has a higher variance of return. Still assuming normality and quadratic utility for the example, risk-averse investors will clearly prefer taking less risk to more risk for the same level of return. This is shown as:

$$Eu\ (w_2 + z\sim) <= Eu\ (w_1 + z\sim)$$

The above graph is an example of a mean-preserving spread (MPS), alluded to in prior papers. The expected mean average return is preserved, but the risk of return has increased. In the typical instance, when the variability or risk of return changes, so too will the rate of return. A greater risk premium will be necessary to induce investors to take greater risk. Thus, the MPS assumption is often considered too strong a requirement for modeling purposes.

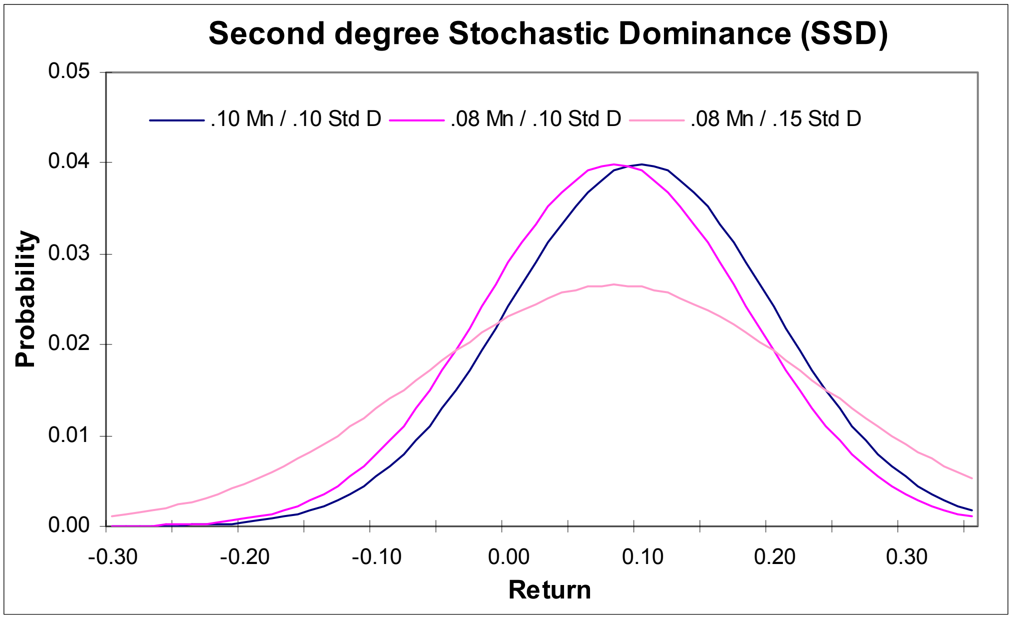
Risk-averse investors will also dislike a lower rate of return for the same level of risk. This is shown in the following example.



Both choices have the same variability of return, with 10% standard deviation, 0 skew, and 0 excess kurtosis, but the second asset now has a lower return than the first asset. When presented with this option, risk-averse agents will prefer the asset with the higher rate of return, 10%, versus the asset with an expected return of 8%. The entire probability distribution has shifted downward for the same level of risk. This is considered to be an example of first-degree stochastic (or random) dominance (FSD) deterioration, which is mathematically noted as:

$$Eu\,(w_2\sim) - Eu\,(w_1\sim) = -\int_a^b u'\,(w)\,((F_2\,(w) - F_1\,(w))\,dw$$

Where, F_i (w) is a continuous cumulative probability distribution. While a higher return is preferred to a lower return, the simple declaration of FSD deterioration will often be insufficient to tell which asset is superior or dominant. In many instances (such as a market-wide downturn), several different return probabilities will have a downward shift in the mean average. In such cases, merely determining a FSD shift will not lead to a selection of assets preferred by the investor / agent, as all of the choices will be FSD deteriorating. The differences between the various FSD shifted probabilities will have to be ascertained. A Second-degree Stochastic Dominance (SSD) deterioration occurs when any form of risk is added to a FSD deterioration.



In addition to the two assets of the prior example, a third asset having the lower rate of return of 8% and a higher standard deviation of 15% is now introduced. Both the second and third assets are said to be FSD dominated by the initial asset having the higher 10% return. But, now we can tell that the third asset is also dominated by the second choice of 8% return and 10% standard deviation. A risk-averse agent will clearly prefer the second choice of the same return and lower variance to the third choice. While an increase in variance is assumed in the example, as a general proposition, an increase in any form of risk would result in a SSD shifted deterioration. The preference for the second asset that has not had further risk added to it is generally noted as:

$$Eu\ (w_3 + z\sim) <= Eu\ (w_2 + z\sim)$$

This short discussion on the nature of risk highlights several important items. First, we started with an exclusive reliance on assets, and yet have concluded that in some cases, the maximization of investor utility cannot be unequivocally determined without looking at liability-related downside risk. Second, we have seen that the classic definition of risk in modern portfolio theory – that of standard deviation – is but one form of risk in a multi-faceted world of risks. Not only are skew and kurtosis relevant, but all random shifts in the probability density function are important, as well. These types of statistical risk are simply not contemplated by MVO. Third, complicated utility equations and concepts are not needed to bring us to this point in the analysis. Stochastic dominance produces general principles applicable to the use of any utility functions, whereby certain economic choices will be preferred over other choices for any utility function employed.

We can still determine asset stream preferences and optimalities by merely making rational investor and risk-averse assumptions, and then determining the FSD and SSD dominant positions. Downside risk is introduced into the process to ascertain which of the comparable SSD assets will be preferred by risk-averse agents. This is consistent with A.D. Roy's safety-first ratio, where MVO derived allocations were established not at the intersection of the Capital Market Line and the frontier, but through a threshold level designed to meet or exceed the stated liabilities.

Utility Functions

As wealth increases, the satisfaction of consuming that wealth increases. The agent is believed to not necessarily want a certain state of wealth, but desires instead the satisfaction (or utility) that consumption of wealth entails. That satisfaction is more formally referred to as utility of wealth. Thus:

$$u(x) \sim \int x$$

or, the utility of wealth, u (x), will be related in some fashion to the wealth itself, defined as x. For every state of wealth, various investors may have different satisfactions of wealth. Utility can be maximized as such:

$$\max E_t\, u(w_{t+1}) \text{ s.t. } w_{t+1} = (1 + R_{p,\, t+1})\, w_t$$

where, $u(w_{t+1})$ is a utility function. The classical view hundreds of years ago was that the value of a lottery should be equal to its mathematical expectations. With such a belief, the utility function would be linear in nature: as wealth increases, utility of wealth increases by exactly the same proportion. Bernoulli showed that the satisfaction of wealth could also be non-liner in nature: As wealth increases, utility of wealth increases, but at a decreasing rate. This type of relationship would appear as a concave curve on a graph depicting wealth on the x axis and utility of wealth on the y-axis (see, utility function graph, below).

The rate of change between wealth and utility of wealth is known as marginal utility, as is considered to be the first derivative of the utility function, styled as u' (x). If utility increases at a decreasing rate whenever wealth increases, marginal utility is decreasing. Most investors are thought to have decreasing marginal utility. More broadly, many micro and macro-economic concepts are based on the law of diminishing returns, which flows directly from diminishing marginal utility.

Utility of wealth can be calculated from many different utility functions. The basic goal is to devise a function that reasonably portrays the investing experience. Harmonic ARA functions (HARA) exhibit linear changes as wealth increases. HARA equations take the general form of:

$$u(z) = \xi\, (n + z / y)^{1-\gamma}$$

Linear equations could also be as simple as u (x) = x, for instance. Another type of function, log-wealth utility, produces a time invariant result, regardless of return predictability or relative risk aversion. Example of log-wealth functions are:

$$u(w) = \ln (w)$$
$$u'(w) = w^{-1}$$
$$u''(w) = - w^{-2}$$

The first derivative will be positive over all values of w, and the second derivative will be negative over all values of w. It is therefore used with more-to-less and risk averse preferences.

Classical utility models used utility equations that were exponential in nature.

$$u (w) = - \exp (-aw) / a$$

The "a" parameter is now a positive scalar. A(w) = a, for all w, so the function is linear and will be CARA. Other forms that have been tried over the years include square root utility, or $u (x) = w^{1/2}$, and combination utility, such as: 1/w + ln (w).

Two major utility functions are the quadratic and the power series. The quadratic is:

$$u (w) = a w – ½ w^2, \text{ for } w <= a$$
$$\text{More generally, } u (w) = aw – bw^2$$

A quadratic was commonly used in the past due to wealth being a function of only the first two moments of the distribution, mean and variance. So it was popular, but was not greatly realistic, since skew and kurtosis were assumed away by the quadratic. It also explains why means-variance optimization is a special case of utility maximization, since it is a special proposition of utility that can only be used when the return probability distribution is normally distributed, or approximately so. Further derivation of the equation leads to:

$$\sigma^2_w = E(w – E(w))^2.$$
$$E (u (w)) = E(w) – b [\sigma_w^2 + (E (w))^2]$$
$$\text{more simply as: } u = E(r) - \lambda \sigma^2$$

The quadratic utility function can be expanded to the familiar two-asset MVO equation of:

$$\sigma^2_{ab} = a^2 \sigma^2_x + b^2 \sigma^2_y + 2 a b \sigma_{xy}.$$

The quadratic is the only utility function that can be stated in direct pricing terms. Since the quadratic is so easy to manipulate into pricing statements, it has become the standard function among many portfolio analysts, from Markowitz forward. It contains some significant theoretical difficulties however. As Eeckhoudt, et al, at 21 stated: "in this case, the EU simplifies to a means-variance approach to decision making under uncertainty. However…. it is very hard to believe that preferences among different lotteries be determined only by the means and variance of these lotteries".

More troubling however, a quadratic has increasing absolute risk aversion by the nature of the equation. It therefore is not in general use today within EU circles. Gollier (2001), at 27, as well as Eeckhoudt (2005) both note this flaw in the quadratic, concluding that the IARA characteristic is incompatible with DARA observations. Gollier (2001) also states that there is no obvious reason why quadratic equations represent the attitude toward risk that agents feel in the real world.

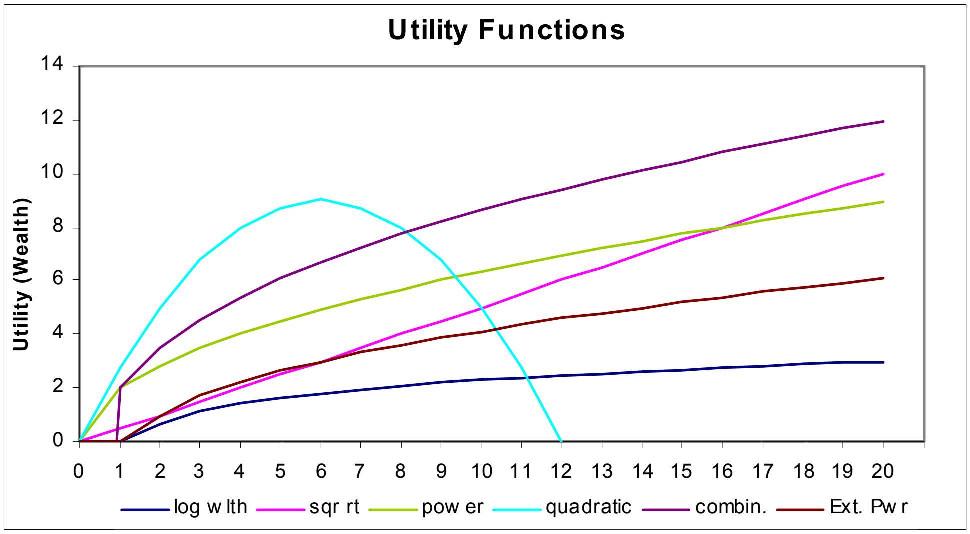
Power utility equations are often used with EU, where:

$$u(w) = w^{(1-\gamma)} / (1-\gamma), \text{ for } w > 0, \text{ with } \gamma >= 0, \gamma \neq 1$$
$$u(w) = \ln (w), \text{ where } \gamma = 1$$

γ is a scalar, and is chosen so that $\gamma > 0$, $\gamma \neq 1$. γ becomes the degree of relative risk aversion (RRA), which will also be discussed immediately below. Asset returns are lognormally distributed. This being the case, it can be shown that a log utility function, $u(w) = \ln (w)$, is a limited form of power utility. Thus, the use of the log function when $\gamma = 1$. Another power equation, extended power, generates a varying RRA depending upon the sign of "a".

$$u(z) = (1 / (b-1)) (a + bz)^{(1-1/b)}$$
$$\text{subject to } z > \max \text{ of } (-a/b; 0)$$

This equation is very useful, since it allows for varying assumptions of risk aversion and conditions of predictability. The power function is currently in widespread use among utility theorists, so much so that it is often treated as the "default" choice for modeling utility. See, Eeckoudt, et al, at 21. The following graph depicts the utility of wealth generated from the same wealth, for the various utility functions described in the text.



As can be seen in the graph, the same, identical wealth generates different utility of wealth, depending upon the chosen utility function. There is no magic all-inclusive measurement of utility, only varying measures of utility of wealth for the varying utility functions. What is considered optimized with one utility equation will definitely not be optimized with another equation. One may now sense the origin of the dispute as to the homogenous assumption of MPT. Efficiency advocates assume investor homogeneity while behavioralists generally believe that investors are heterogeneous in many of their investment preferences, including choice of utility function and type of risk aversion. The search for optimality across time, or at any one time, should therefore concentrate on which utility function most appropriately models the economic behavior of investors.

Risk Aversion

The basic concept of risk aversion. An investor could be risk averse, risk neutral, or risk taking with regard to taking a fair gamble. The risk averse agent will always prefer the utility of the lottery with certainty rather than the lottery itself. This means that any risk averse agent will have a utility function that is concave, with E(w) <= u (w), with u (w) on the y axis and w on the x axis. Risk aversion occurs where an investor rejects a fair gamble because the investor feels that the disutility of a loss is greater than the utility of an equivalent gain. Most investors are thought to be risk averse, and empirical evidence tends to support the proposition. An agent is risk averse if he dislikes all zero-mean risk at all wealth levels.

If an agent is risk neutral, a linear utility will be exhibited, with the lottery being equal with its expected payoff. For risk takers, the situation will be reversed, so that a convex curve describes the agent's preferences. The agent will take a fair gamble.

Investors who can state their utility preferences towards a fair gamble will then be able to limit and concentrate on their universe of risky investments they are interested in. Risk averse investors will only choose the efficient frontier above the point of minimum variance when choosing between portfolios. Risk takers may choose a leveraged position above the tangency point between the frontier and the CML, for example.

The Risk Premium and the Arrow-Pratt Approximation. A risk-averse person or agent may still accept risk if the rewards are great enough. For example, the investor may be interested in an equity so long as the return from that equity exceeds a risk free asset by sufficient amounts. The amount that makes an agent indifferent to the two choices of taking the risk versus paying to avoid the risk is the certainty equivalent, and is:

$$E\,u\,(w + z\sim) = u\,(w + e)$$

Where, e = the certainty equivalent. The risk premium is considered to be the amount of reward or return that is offered over the risk free rate in order to induce a person to accept the risk or uncertainty of return. It is the asset price less the RFRR, but can be more formally stated in terms of utility as:

$$E\,u\,(w + z\sim) = u\,(w - \Pi)$$

where, the risk premium is designated as Π. The risk premium will be non-negative when u is concave, and will be zero when u is linear. It will be non-positive when u is convex. Arrow and Pratt established many years ago that, for small risks, the risk premium is approximately proportional to the variance of its returns. The premium necessary to induce agents to accept risk becomes a simple function of mean and variance. The Arrow-Pratt Approximation of Risk Premium:

$$\Pi = \tfrac{1}{2}\,\sigma^2\,A(w)$$
$$\text{Then, } \Pi\,(k) = \tfrac{1}{2}\,k^2\,\sigma^2\,A(w)$$

Since only the first two moments of the probability distribution are made part of this equation, the Approximation holds only when returns are normal, or approximately so. Where the probability distribution becomes skewed to the right or left of a typical bell-shaped curve, or exhibits positive or negative excess kurtosis (becomes heightened or flattened compared to a bell-shape), then the third and fourth moments of the distribution of returns become important. Additionally, the Arrow-Pratt approximation is applicable to a culmination of small risks. Where large risks are involved however, the risk premium may be dependent upon skew and kurtosis, thereby negating the means-variance framework. Eeckhoudt, et al (2005), at 35.

Additionally, for large risks, the approximation will underestimate the risk premium. This is shown in a nice graph in Gollier (2001), at 23. The degree of skew or kurtosis would affect the desirability of taking the risk, resulting in a different risk premium being paid for two different distributions having the same mean and variance. This again demonstrates that the means-variance criterion is a special case of the more general proposition of expected utility, applicable only when the return probability distribution is approximately normal in shape.

Notice that the proposition is stated in individual terms - the degree of each agent's risk aversion is important. Individuals may have different attitudes and thoughts about risk, return, and the degree of risk aversion. This leads to heterogeneous expectations by investors, and not necessarily to the homogenous investor assumption of MPT.

Absolute Risk Aversion (ARA). Different agents will have differing amounts of risk aversion. In more mathematical terms, absolute risk aversion measures the degree of concavity in utility graphs, which is calculated as the rate at which marginal utility is decreasing. ARA is measured in units of wealth, as varying monetary amounts will be necessary to induce people to take risk. A (w) measures the maximum amount that an agent will pay to get rid of a small risk. The applicable equation is:

$$A(w) = - u''(w) / u'(w)$$

For power utility series, ARA becomes the following (From, Stangeland and Turtle (1999)):

$$r_A(z) = - u''(z) / u'(z) = 1 / (a + b z)$$
$$\text{and, } u(z) = 1 / (b-1) (a + b z)^{(1-1/b)}$$

where $b > 0$, and $z > \max [- a / b, 0]$. Taking the 2^{nd} derivative with regard to utility of z produces the negative of the ARA equation. The reciprocal of risk aversion is referred to as risk tolerance, and is mathematically stated as:

$$T(z) = 1 / (A(z) = - u'(z) / u''(z)$$

When absolute risk aversion is concave, for instance, risk tolerance will be convex.

Most researchers believe that individuals are generally decreasing in their absolute risk aversion (DARA). An agent will decrease his or her absolute risk aversion if the risk premium decreases as wealth increases. This will occur whenever the utility function produces a concave curve. With a risk averse agent, A will be positive. It will be zero with risk-neutral preferences and negative with risk takers.

Relative Risk Aversion (RRA). This is simply a unit-free designation of ARA, with RRA being the percentage at which marginal utility decreases for a percentage decrease in wealth. While ARA is not unit free and is measured per unit of money, relative risk aversion is unit free, essentially being the wealth elasticity of marginal utility. The relative risk aversion equation is:

$$R(w) = - w u''(w) / u'(w) = w A(w)$$

If the above power utility functions are in use, then relative risk aversion is:

$$r_A(z) = u''(z) z / u'(z) = z / (a + b z)$$

While there is a general consensus that investors are risk averse and also are decreasing in their absolute risk aversion, there is far less agreement on whether investors are decreasing in their relative risk. Constant relative risk aversion (CRRA) is often assumed for purposes of convenience. Elton, Gruber, et al, states that CRRA is often assumed, although the justification may be more "convenience" than "descriptive accuracy". The empirical evidence is contradictory on whether relative risk is increasing or decreasing. See, Eeckhoudt, at 22; Elton, Gruber, et al, at 222.

The lack of consensus as to what type of relative risk investors possess is in large part due to the contradictory risks that are being balanced in the relative risk formulation. Under a DRRA assumption, investors are more risk-tolerant and less risk-averse as they become wealthier. This effectively reduces the risk premium for these types of investors. But it also means that the size of absolute risk increases. This in turn increases the risk premium.

With constant relative risk (CRRA), the two effects cancel each out exactly – this was the basis for the time irrelevance belief in portfolio theory. There is no particular reason to believe, however, that one effect will dominate over the other as wealth levels change. With two contradictory effects involved, there will be no clear indication as to which effect will dominate. Investor preferences will ultimately determine the direction of relative risk. For instance, one study indicated that investors with tolerances for very large losses would invest in risky assets in the long-term but not necessarily in shorter time periods. See, Ross (1999).

The importance of the relative risk determination is that asset allocation largely turns on relative risk notions. With decreasing relative risk, investors will be willing to take more risk as wealth increases. This leads to a higher equity allocation. With a constant relative risk, investors will not change their allocations in response to a change in wealth. With increasing relative risk, investors will become more conservative with increases in wealth, thereby decreasing equity allocations.

Many financial practitioners effectively assume decreasing relative risk early in an investor's life. Increasing risk aversion may be assumed as one moves into a consolidation phase, followed by a constant to increasing risk aversion at retirement. This generates increasing equity allocations early in life, but then decreasing equity allocations as investors begin planning for retirement. A constant equity allocation may be appropriate once an investor achieves retirement, although if assets significantly exceed liabilities, increasing allocations could be used for the "surplus" assets. This pattern is quite consistent with pricing variance changes liability issues across varying time horizons. As time frames lengthen, pricing variance is remarkably reduced, allowing investors to increase their exposure to equities without incurring more pricing risk. As wealth accumulates however, investors will become more concerned with capital preservation and maintenance issues, and this gives rise to a more cautious attitude towards equities in contemplation of near-term retirement needs. At retirement, investors will want stability of income, and may therefore find themselves drawn to fixed income assets and income-oriented equities. This results in a more mature and stable view towards asset allocation, so as to avoid current-year income fluctuations.

Briefly reviewing the major utility functions and impact on risk aversion is very useful. This summary is from Kritzman and Rich (1998), and also to some extent, Stangeland and Turtle (1999). Extended power functions will generate DARA, and varying relative risk, depending upon parameter settings. The quadratic equation exhibits both absolute and relative risk aversion increases (IARA and IRRA) as wealth increases. The function used by the classical theorists many years ago, log-wealth utility, is DARA in the absolute risk aversion but will produce no change in utility with constant relative risk aversion. The square root function is less risk averse than log wealth, and the power function is more risk averse than the log wealth. Combination utility can generate DARA and DRRA.

Static Choices

One Risky Asset. Now that some basics have bee sketched out, we can turn our attention to investment choices. The analysis starts off with simple static equations, with dynamic processes developing from there.

An uncertain environment can be described by listing all possible outcomes, with X being the set of all possible outcomes. $X \equiv (x_s)_{s=1 \ldots s^*}$. The probabilities of the outcomes are also needed; $p_s >= 0$. This generates consumption choices under conditions of certainty. The investor's gamble can be expressed as $H_1 = \sum_w P_1 h_1$.

Additional lotteries are assumed to produce preferences as to ordering that are independent of the additional lotteries used. This assumption takes consumption choices under certainty into decision-making under conditions of uncertainty, and this develops into the EU theorem of von Neumann and Morgenstern. L^a has a preference function that is linear with $L^b \Leftrightarrow \Sigma\, p_s^a u_s >= \Sigma\, p_s^b u_s$. Thus, the expected utility is defined as:

$$E(u) = \sum_w u(w)\, p(w)$$

The impact from risk depends upon the risk itself, the wealth level, and the utility function. When risk z~ has an expectation differing from zero, the certainty equivalent, of risk z~ is the definite increase in wealth that has the same effect on welfare as bearing the risk z~. The amount that makes an agent indifferent to the two choices of taking the risk versus paying to avoid the risk is the certainty equivalent. This is stated as:

$$E\, u(w + z\sim) = u(w + e)$$

Investor choices can be stated in terms of equities versus risk free assets. Assume, a two asset class universe of only stocks and risk free assets; w_o is initial wealth; r is rfrr of bond; return of stocks is x~; α is amount (not %) of stocks; thus, $w_o - \alpha$ = amount of bonds. Then:

$$y\sim = x\sim - r,$$

This is the excess return of the risky asset, which is also known as the risk premium. This is a one period model, so we further assume that the agent will consume all wealth at end period 1, which will then be $w + \alpha\, y\sim$. The investor will choose the amount of stocks to maximize EU such that:

$$\alpha^* \in \arg\max_{\alpha} Eu\,(w_o + \alpha\, y\sim).$$

This equation is considered the standard portfolio problem. Translated to a sentence structure, the expected utility will be maximized by initial wealth levels plus the amount of the risky asset multiplied by the excess return. Flowing logically from this statement and equation, the optimal amount of the risky asset will be positive IIF the excess return is positive. (Gollier, 2001, at 54, states this as a proposition). If $y\sim = 0$, then $\alpha^* = 0$.

<u>The impact of risk aversion.</u> So long as u(x) is differentiable, as is the normal case, α^* will be reduced when risk aversion is increased. α^* will increase if absolute risk aversion is decreasing. Demand for the risky asset will be positive as soon as there is a positive risk premium. Conversely, if relative risk aversion decreases, the share of the risk asset would increase at optimality. This only makes common sense, as a higher risk aversion suggests a higher degree of concavity of u(x). The approximation for the percentage of wealth invested in stocks is:

$$(\alpha^* / w) \cong (\mu_{y\sim} / \sigma^2_{y\sim}) * (1 / R(w))$$

where R(w) is the degree of RRA at wealth state w, and u and variance are the mean and variance of the excess stock return. Using historical data from Shiller, Eeckhoudt (2005) then calculates the appropriate equity allocation. If R = 2, the approximation would generate 117% equities, based on historical data. This is unreasonably high, leading to the equity premium puzzle and a search for other utility factors (such as background risk, liquidity constraints, etc) that might reduce the optimal % of risky assets in an expanded utility equation using these factors.

(KCK Note: Using Ibbotson one year historical data from the Wilshire and Lehman, I found the optimal risky asset to be 99.7%, based on equity mean average return of 12.77% and standard deviation of 21.27%, RFRR of 3.744%, and R = 2. This is consistent with the Eeckhoudt example, above. This also suggests a very high equity percentage considering that MVO on a one year horizon, without any considerations of RRA, generates 30.3% equities. R would need to be 6.5 or so to get to 30% equity allocation, which would be a very high RRA).

Generally, when the agent is CRRA, optimality occurs for a fixed share of stocks, and demand for equities is proportional to wealth: $\alpha^*(w) = k\, w$. Gollier (2001, at 58) also states that when u(x) is a HARA function, there is a linear relation between the optimal exposure to risk and the wealth level. For CARA functions, α^* is independent of wealth.

For CRRA functions, α^* is proportional to wealth, with the percentage of risky assets remaining fixed independent of wealth (and thus the amount of risky assets increase linearly or proportionally to changes in wealth with the same equity % maintained).

Two Risky Assets. Adding uncertainty to the pricing of the risky asset always lowers the agents' expected utility. More accurately, risk averse agents dislike adding zero mean noise (i.e. price uncertainty) to the possible outcome of wealth. This is stated mathematically as:

$$Eu(w\sim_2) = \sum_{s=1}^{n} p_s \, Eu\,(w_s + \varepsilon\sim_s) <= \sum_{s=1}^{n} p_s \, u\,(w_s) = Eu\,(w\sim_1)$$

Where, the w_1 is the wealth state of the risk free and risky asset combination, and w_2 is the wealth of the risk free and the risky asset having price uncertainty. P_s is the probability of the event, s, and $\varepsilon\sim_s$ is the price uncertainty or noise being introduced. Thus, $w_s + \varepsilon\sim_s$ is the lottery in which some of the outcomes are themselves lotteries having uncertain pricing. The expected value of $\varepsilon\sim_s = 0$, so we are still at zero-mean risks. The above equation holds that the probability weighted expected utility of wealth plus the uncertainty of wealth will be < = the probability weighted utility of wealth. Agents will always prefer wealth (or the utility of the wealth) versus the same wealth (or utility of wealth) that has pricing uncertainty attached to it. Risk averse agents will always prefer certainty of wealth over uncertainty of wealth, even if the wealth outcome is the same either way.

This situation has maintained the same mean of the probability distribution with either w_1 or w_2 wealth states. The shape of the distribution changes by adding noise, but the mean does not. This is referred to as "mean preserving spread" in the distribution, or MPS. In continuous cases, risk can also be increased where an integral condition is equivalent to a sequence of mean-preserving spreads. With MPS, the mean does not change, but pricing volatility may increase, for instance.

Two Risky Asset Portfolio. Assuming the two assets both have the same distribution of $x\sim_1$ and $x\sim_2$, and they are IID. The optimal portfolio will be:

$$\max_\alpha Eu\,(\alpha\, x\sim_1 + (w - \alpha)\, x\sim_2\,)$$

Now, α is the amount invested in the first risky asset. The function will be concave in the decision variable. Solving for the first order condition (setting the 1^{st} derivative to 0) yields:

$$E(x\sim_1 - x\sim_2)\, u'(\alpha^* x\sim_1 + (w - \alpha^*)\, x\sim_2) = 0$$

Because $x\sim_1$ and $x\sim_2$ are IID, they can be added together:

$$E\, x\sim_1 u'\,(\,\tfrac{1}{2}\, w\,(x\sim_1 + x\sim_2) = E\, x\sim_2 u'\,(\tfrac{1}{2}\, w\,(x\sim_1 + x\sim_2))$$
$$\alpha^* = \tfrac{1}{2}\, w.$$

This is an example of risk diversification for two assets having the same return distribution. All other portfolios will be 2nd order dominated by this one, and rejected by risk averse investors.

Within the means-variance case, risk can be reduced through diversification of assets. This works well where there is a normal distribution and CRRA. The two-asset means variance equation reduces to:

$$a^*_1 = (1 / A)*[(\mu_I – x_0) / \sigma_{ii})]$$

where a* is a vector of optimal shares in the risky assets, and A is an index of ARA of the investor. All investors should purchase the same portfolio of risky assets, under this solution. Risk aversion allows one to then substitute the rfrr into the portfolio. The single market portfolio conclusion is known as the two-fund separation theorem of portfolio theory, with the market portfolio effectively comprising an index mutual fund. This conclusion relies on two assumptions however: 1) the markets are informationally efficient; 2) investors have means-variance preferences.

Note that the two-asset problem so far has generated the same result as with MVO.

From Gollier, 2001, at 141. How does two endogenous risks to a portfolio affect optimality? There is no obvious reason why such risks would be complementary or substituting in nature. As seen above, if the two risks are IID there will be a diversification effect whereby the portfolio will be split into two equal parts, lessening the demand for the initial risky asset. But, the diversified portfolio will be attractive to investors versus the RFRR, since the risk of the portfolio is now comparatively reduced. This will increase the demand for both risky endogenous assets in the portfolio, reducing the exposure to the RFRR. These two effects essentially work in opposite directions.

There is also a wealth effect and a pure risk effect. The wealth effect is from new wealth generated through exposure to the additional risky assets with a higher return. With DARA, this effect increases the demand for all risky assets. This would suggest that both risky assets are complements. But, a zero-mean, pure risk to background wealth also exists. This has a tempering effect if the agent is risk vulnerable. The pure risk effect will act as a substitute with independent risks.

The general case of two risky assets (x and y) is as follows:

$$g (\alpha_x , \alpha_y) = Eu (z + \alpha_x x\sim + \alpha_y y\sim)$$

<u>The Arrow-Debreu Economy.</u> The model will now be expanded to include all investment choices. This is a complete market, including options, hedges and insurance. Arrow (1953) and Debreu (1959) were the first to examine this. Assume S possible states of nature, s =0….S-1. Probability of s is: p_s. Arrow-Debreau securities are those assets providing pay-offs IFF state s occurs, and 0 otherwise. Buying an Arrow-Debreau security is equivalent to betting on that state. *(KCK note: notice that the entire*

probability distribution, presumably with variance, skew, and kurtosis, is now in the equation through the probability statement).

If there are two states of nature (S=2) and a risky asset whose initial price is unity, then $P_0 - 1$ is state 0 and $P_1 - 1$ is state 1. So, P_s , s = 0, 1. If rfrr has return r in both states, then markets complete. Assume, $P_0 < 1+r < P_1$. Thus, the risky asset returns both more and less at times than the rfrr. This is the equilibrium condition for 2 states. We can replicate the A-D security at state 1 by purchasing α units of the risky asset and borrowing B units of the rfrr, such that:

$$0 = \alpha \, P_0 - (1 + r)\, B$$
$$1 = \alpha \, P_1 - (1 + r)\, B$$

This states that in state 0, revenue is 0, while in state 1, revenue is 1. Markets will complete if there are as many assets whose vectors of state-contingent payoffs are linearly dependent as there are number of states. Now, assume Π_1 = price of an A-D security with state s = 1. This is the price to be paid IFF state 1 occurs, and is known as the "state price". The price can be deduced from real assets upon which contingent claims can be duplicated. The price of A-D security in state 1 can now be said to be:

$$\Pi_1 = \alpha - B = (1 - (P_0 / (1 + r))) / (P_1 - P_0)$$

Now that we know the price of the A-D security, we can find the price of any financial asset. The price of a bond will be:

$$P_B = (1 + r)^{-1}$$

For a portfolio having a rf payout of 1, and without arbitrage:

$$\Sigma^{S-1}{}_{s=0}\, \Pi_s = P_B = (1+r)^{-1}$$
$$p_s = \Pi_s\, (1+r)$$

p_s is considered the risk-neutral probability for state s. The price of any asset in the complete system is merely the expected value of the asset, discounted at the rfrr, with the expectation of risk-neutral probability rather than the true probabilities. For state-contingent payoffs of $y_0 \dots y_{s-1}$, and with no arbitrage, the price must be:

$$P = \Sigma^{S-1}{}_{s=0}\, \Pi_s\, y_s = \Sigma^{S-1}{}_{s=0}\, P_s\, y_s / (1+r) = E\, y\sim / (1+r)$$

This would be the risk-neutral probability distribution of the price of asset y. *KCK: Notice the similarity of the PV formula for an asset: $PV_y = FCF_y / (1+r)$. And, $P_y = E(y) / (1+r)$.* Let us now select a portfolio of A-D securities maximizing expected utility under a budget constraint. Let c_s = investment of state s, and w = initial wealth. The problem would be to:

$$\max_{c.}\ \Sigma\, p_s\, u\, (c_s), \text{ subject to } \Sigma\, p_s\, \Pi_s\, c_s = w$$

with p_s being the probability of state x. $\pi_s = \Pi_s / p_s$, so by substitution:

$$\max_{c0.....cs\text{-}s} \Sigma^{S-1}_{s=0}\, p_s\, u\,(c_s) \text{ s.t. } \Sigma^{S-1}_{s=0}\, \pi_s\, c_s = w$$

This equation is more flexible than in the one risky vs rfrr model, where c_s = w + alpha s. The optimal risk exposure $c_0.....c_{s-1}$ is now constrained only by the budget constraint of $\Sigma^{S-1}_{s=0}\, \Pi_s\, c_s = w$. This A-D format will still lead back to the standard portfolio problem, discussed above, if the further assumption is imposed that agents are constrained to linear relations between wealth and market return. Thus, this formulation is merely a broader statement of the standard portfolio problem that can be applied to all securities, instead of a risky vs rfrr. The problem can be generalized to:

$$\max Eu\,(c(x\sim) \text{ s.t. } E\,\pi\,(x\sim)\, c\,(x\sim) = w$$

Optimality is now determined by the ratio of price to probability, $\pi_s = \Pi_s / p_s$. When the ratios are different, we would expect the investor to reduce demand for those assets whose prices are large relative to the probabilities *(KCK: ie. overvaluation situations).* The FOC is both necessary and sufficient for optimality. $u'\,(c^*_s) = \varepsilon\, \Pi_s / p_s$., which is also: $u'\,(c^*_s) = \varepsilon\, \pi_s$.

An increase in risk aversion under the standard problem would always reduce exposure to risky assets. The situation is now more involved, with a global measure of absolute risk being used. The FOC implies a different level of consumption for each value of π_s. u' would be decreasing with regard to risk aversion, as less is consumed in more expensive states. It could be noted as:

$$c^*_s = C(\pi_s), \text{ for all } s,$$

with C = consumption or purchase of asset. Taking the FOC of u' (C), the inverse of Arrow-Pratt is arrived at: u'(C) / u''(C). This is called the local measure of absolute risk aversion, - C', and is the amount of risk the agent is willing to take. This can be stated as the optimal exposure to risk:

$$C'\,(\pi) = -\,T\,(C\,(\pi)) / \pi.$$

With this equation, more risk tolerant agents should purchase a riskier portfolio. The absolute risk tolerance is: $T(C) = -\,u'(C) / u''(C)$.

Now, the consumer's goals can be maximized, subject to a budget constraint. Building this to a graph depicting the investment choices, let's assume that the rfrr, r = 0. An indifference curve can be sketched out between two risky choices, with the equation of:

$$p_0\, u(c_0) + p_1\, u(c_1) = k,$$

with k being a constant. An infinite series of utility curves can be sketched out, offering ever-expanding pairings of “indifference” between the two consumption choices. The budget constraint is:

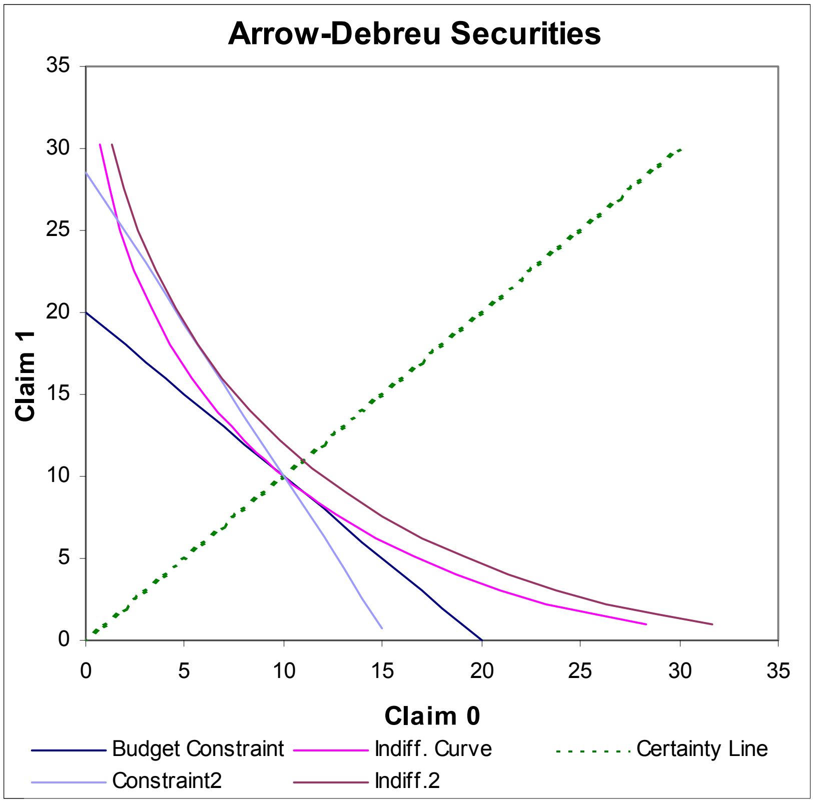
$$p_0 c_0 + p_1 c_1 = w.$$

(c_0, c_1) defines the set of claims for which EU is constant in state-claims space. The slope of the indifference curve can be found through implicit differentiation. The absolute value of $\partial c_1 / dc_0$ is the marginal rate of substitution (MRS) between states. It is the trade-off at the margin between one unit of consumption state 0 and state 1. This is the second order risk aversion under EU. It implies that the indifference curve is convex.

Assuming that the state prices = state probabilities, the equilibrium investments in c_0 and c_1 would then be the tangency between the budget constraint and the indifference curve. The 45 degree line (in green below) is known as the certainty line, as this is the equilibrium point for c_0 and c_1 for any given level of wealth. A new indifferent curve with a higher k could be sketched out for a higher level of wealth (and a greater budget constraint). The equilibrium level of consumptions would follow the certainty line. The graph below details the budget constraint, indifference curve, certainty line, and point of optimality.

If the price of claim 0 becomes greater than the probability of state 0, that would rotate the budget constraint toward c_1, since the price of claim 1 now becomes lower than the probability of state 1. The constraint now would be: $\Pi_0 c_0 + \Pi_1 c_1 = w$, as the probabilities are replaced by the pricing of claims. This leads to a new equilibrium point with more claim 1 and less claim 0. The rotation of the budget constraint and the new equilibrium point is shown below.

(KCK note: The choices between two assets become very similar to some curves in Public Finance dealing with a current consumption / deferred consumption analogy. Optimal portfolio selection for two assets in a single period model is virtually identical to a two-consumption choice of households. This leads to a representative agent at the macro level, discussed briefly at the end of this paper).



Multi-period models. The discussion will now be extended to more than one time period. Assume that agents can postpone risk to the future and can defer gains and losses over several periods. This can provide strategic advantages. Time is denoted as n dates, t=1….n-1. y_t is revenue received with certainty at date t. r =- rfrr, for borrowing and lending. c is once again consumption. The budget constraint is now:

$$z_{t+1} = (1+r)\,(z_t + y_t - c_t)$$

with z_t being the cash transferred from date t-1 to date t. Initial cash is zero. This becomes a lifetime consumption model *(KCK: similar to public finance models)* with the different dates of consumption sketched out on the indifference curves, c_0 and c_1, for example, now being expressed as consecutive dates. The no debt constraint of $z_n >=0$ is imposed so that agents will not die in a debt position. We assume certainty regarding wealth and income. The PV of the lifetime budget constraint cannot exceed the PV of the lifetime income, so:

$$\sum^{n-1}_{t=0} \Pi_t\, c_t <= w_0.$$

Where, $\Pi_t = (1+r)^{-t}$ and w_0 = lifetime wealth. Π_t is the price of the zero coupon bond with maturity at date t. The budget constraint can be then be shortened to: $\sum^{n-1}_{t=0} \Pi_t c_t <= w_0$.

The agent will attempt to choose an optimal consumption path across time. The lifetime consumption flows are $U(c_0, \ldots c_{n-1})$, and the indifference curve is now generally defined as $u(c_o, c_{n-1}) = k$. We assume an increasing and concave utility function. Optimal consumption is:

$$\max_c u(c_o \ldots c_{n-1}) \text{ s.t. } z_{t+1} = (1+r)(z_t + y_t - c_t)$$

The intertemporal optimization is the same as the Arrow-Debreu consumption equilibrium of two states in a single-period model, only now optimizing total consumption across two periods. Instead of states of nature, we have dates. Probabilities are replaced with discount rates and, ultimately, prices based upon discount factors. The Arrow-Debreu securities are now replaced with the more realistic zero coupon bonds. See Eeckhoundt, et al, at 92, for more discussion on the equivalence between consumption in a single period model versus lifetime consumption models.

For multi-period models, an independence of consumption has to be assumed. This stops the formation of consumption habits. The added assumption implies that utility functions in any one time period can be separated, so long as the total utility across all time periods remains the same. $U(c) = \sum u_t(c_t)$. Then, $u_t = p_t u(.)$, and the felicity functions (to distinguish from u(c)) are proportional to each other: $ut(.) = p_t u(.)$. p_t is now considered a "discount" factor for felicity $u(c_t)$ occurring at date t. p really becomes a preference factor at the individual level expressing the desire to consume now or later (so do not confuse the use of probability, p, from the single-time model). The model now becomes:

$$\max_c \sum^{n-1}_{t=0} p_t u(c_t) \text{ s.t. } \sum^{n-1}_{t=0} \Pi_t c_t = w_0.$$

The FOC is: $p_t u'(c_t) = \varepsilon \Pi_t$, for $t = 0, \ldots n-1$. ε is a lagrangian multiplier. This represents the marginal lifetime utility of an increase in the PV of wealth. Since, $\pi_s = \Pi_s / p_s$, the FOC and optimality becomes:

$$c_s = C(\Pi_s / p_s),$$

whose function C satisfies: $u'(C(\pi)) = \varepsilon \pi$. The FOC is now the point of solution or optimality of the portfolio. Investment choices can be graphed out the same as above, only now with consumption dates forming the y and x axis instead of claims on different consumption states or items.

When time periods are made part of the process, the basic choice of immediate consumption versus deferred consumption becomes more evident than with analysis conducted through modern portfolio theory. This represents a significant advantage over a traditional, unitary time model. The CRRA feature was deliberately chosen in the above calculations to show that even when classical relative risk aversion preferences are selected, time horizons are quite relevant in portfolio analysis. In the situation above, the greater preference for current consumption shifted the point of optimality off of the 45 degree certainty line.

Consumption smoothing. This brings us to another closely aligned item of public finance, that of smoothing of consumption. The preference or felicity function u_t can be seen as an aversion to consumption fluctuation across time. Since marginal utility is decreasing with regard to consumption, the agent will have incentives to smooth consumption over time. The FOC will be $u'(ct) = \varepsilon$, for each period, so that the optimal consumption path does not fluctuate between periods. If income fluctuates at any one time, optimality would change: the agent will save or borrow more money to maintain the same consumption. Lifetime utility will be maximized by consuming the same in each time period. This is the case so long as the felicity functions are concave, with declining marginal utility and decreasing absolute risk aversion.

Changing consumption across periods would entail giving more than one unit of consumption tomorrow for the loss of one unit of consumption today. Assuming no credit market, so that $c_t = y_t$, the marginal cost or benefit of a consumption transfer across periods can be stated as:

$$u' (y_0) = (1 + k)\, u' (y_1).$$

k becomes the marginal loss in utility of giving up one unit of consumption today for 1+k units in period 1. The resistance to substitution over time is roughly proportional to the growth rate of consumption. This is a relative aversion to consumption fluctuations over time, and is comparable to RRA.

Optimal consumption growth. The real interest rate up to now has been assumed to be zero. Now, assume a constant rate per period for discounting purposes (this would be an exponential absolute growth, and constant growth year to year). This will be: $B = (1 + \delta)^{-1}$. Rearranging, $\delta = (1-B) / B$, and $\delta > 0$. B is simply a scalar. A person may also be impatient, and this can be expressed in the δ variable. A real interest rate coupled with impatience may lead agents to not smooth completely across time. Higher amounts of impatience will lead to current consumption preferences and lower forward consumption paths. A higher interest rate makes savings and less current consumption attractive.

The two factors cancel out with a CRRA assumption. When the discount is exponential and CRRA exists, it is optimal for consumers to let consumption grow at a constant rate. $C_t = c_0 a^t$, or initial consumption is set at some level that satisfies the lifetime budget constraint. This is expressed as: $\delta = r = 0$.

Business cycles force fluctuations in consumption. But the overall impact upon state welfare may be small (Lucas, 1987), due to consumers having extremely low aversion to small fluctuations in consumption. The argument is to concern oneself with LT growth rate rather than the reduction of volatility of the business cycle.

If savings dominate impatience, then the growth rate will increase in later years, and decline in current time frames, as the agent attempts to maximize savings. If savings is low relative to impatience, then $\delta > r$, and the agent will consume more today.

Uncertainty of Labor Income. Assuming a two period model with definite income y_0 but unsure income in y_1, we also assume the risk is exogenous. Agents will save in time 0 to maximize expected lifetime utility. The optimal saving under uncertainty is s*.

$$\max_s V(s) = u_0 (y_0 - s) + Eu ((1 + r) s + y_1\sim)$$

The FOC is: $u'_0 (y_0 - s^*) = (1 + r) Eu' ((1 + r) s + y_1\sim)$. The willingness to save is determined by the expected marginal utility of future consumption. The incentive to save is heightened by the uncertainty regarding future incomes. This is known as a precautionary effect, being based on prudent behavior, and is also known as prudence (which we covered briefly above). Precautionary savings will be positive for all possible distributions of future risk, where the investor has a normal (risk averse) marginal utility function.

The intensity of precautionary savings can be estimated. The precautionary premium will be positive whenever the agent is prudent, or $u''' > 0$. The precautionary risk premium will be similar to the Pratt-Arrow Pratt risk premium: $\psi = ½ P (w_0 + Ey\sim_1)\sigma^2_{y\sim 1}$. P is now the degree of absolute prudence. The precautionary premium ψ will be decreasing in wealth if P is decreasing in wealth.

With a constant risk free rate of savings, $\delta > r$, the optimal savings will be zero, or $s^* = 0$. The consumer will consume current income in each period, with no savings. If $y\sim_1$ now becomes risky, $s^* = 1/2\ \psi$, solving from the above Prudence equation. If the agent is prudent, there will now be a precautionary savings, s* > 0. The more intense the prudence, the more savings, s*.

Uncertain savings. Now let's reverse the situation above, and assume a known labor income but with unknown savings rate of return. Assume a risky rate of return of 1+ r~, Er~ > 0. Assume a two period model. If the expected rate of savings $_0$ = discount preference, $\delta > r$, consumption will be equal between periods. $c_0 = c_1$. Adding savings risk to the equations will be inconclusive, as savings is less attractive than a rfrr with the same average return. Conversely, period 1 will have a precautionary effect to the prudent investors.

There must be a relatively high level of prudence in order to have the precautionary effect dominate. Relative prudence is absolute P (z) * wealth level, z. s* will increase if relative prudence > 2; stay the same if relative P = 2; and decrease if relative P < 2. For felicity functions and CRRA investors, s* will increase if RRA > 1; stay the same if RRA = 1; and decrease if RRA < 1. There has been an inability in the literature to clearly distinguish between risk aversion and fluctuation aversion.

Exogenous / Background Risk. External, macro level risks can be introduced, with a zero mean:

$$\alpha^{**} \in \arg\max_{\alpha} Eu\,(w + \varepsilon\sim + \alpha\, y\sim).$$

We assume that $\varepsilon\sim$ is not correlated with portfolio risk. In the above equation, the question simply becomes whether the zero-mean risk makes people more averse towards independent risks.

Gollier (2001) states the issue as: $\max_{\alpha} Eu\,(w + \alpha\, y\sim + x\sim)$, where x~ is the background risk. The return of the risky asset, y~ is assumed to be independent of x~. The decision of α is typically done prior to knowledge of both risks, and thus optimality may be affected.

Gollier (2001), at 115-117, shows that DARA is preserved by the introduction of independent background risk. If x~ and y~ are independent, then an increase in risk aversion reduces the optimal exposure to risky assets so long as u exhibits DARA. If the two risks are negatively correlated, then background risk will generate a partial offset to portfolio risk. If the risks are positively correlated, the reduction in risky assets is accelerated from the introduction of noise. Thus, an increase in risk aversion in the presence of exogenous noise does not necessarily imply a reduction in portfolio risk. It depends on whether the two risks are independent, positively or negatively correlated.

Generally, the introduction of background risk affects the agent's behavior towards risk. Assuming u(x) is concave and the investor has DARA preferences, it is still true however that he will take more risk if his wealth is increased.

(From Gollier (2001, at 127, et seq)): The existence of background risk can increase aversion to other independent risks. Risk aversion is thus vulnerable to universally undesirable background risk in some situations. This occurs where a reduction in wealth raises risk aversion, and the existence of background risk raises risk aversion as well. DARA is necessary for risk vulnerability to exist, since ARA must be convex and decreasing. The presence of "unfair" background risk will thus raise aversion to other independent risks, such as portfolio risk, where DARA exists. Adding noise to the background risk will then increase the aversion to the other independent risks. This means that in the standard portfolio problem, adding a zero-mean noise to the risky asset return distribution will reduce the demand for it.

Any background risk with a nonpositive mean will raise the equity premium that an agent is ready to pay to avoid another independent risk. Such background risk will never convert another undesirable risk into a desirable one.

It is widely believed that the presence of risk in background wealth has an adverse effect on the demand for other independent risks. As noted above, this is not true in all instances. For instance, the existence of background risk can substitute for other independent risks. Ross (1999) suggests that two separately undesirable risks can still be jointly desirable, and are therefore complementary in nature. This would occur where negative correlation exists between the two undesirable risks.

As a summation on noise: the effect that noise or other macro risks have on portfolio composition really depends upon whether the exogenous risk is independent or correlated with portfolio risk. Assuming DARA and independence of risks, exposure to risky assets will decrease in the presence of background risk. If the risks are positively correlated, this decrease to risky assets will actually accelerate. If negatively correlated, there will be a compensating effect so that portfolio risk could actually be dampened by the introduction of noise. Generally, many investors would decrease the equity portion of a portfolio if faced with the introduction of exogenous risk from outside of the portfolio. This implies that a representative agent may believe that noise and pure portfolio risks are either independent or positively correlated, in many instances. The equity premium would increase with such a supposition, as it would take more money to induce the acceptance of endogenous portfolio risk as well as exogenous risk. Otherwise, the RFRR could always be defaulted to.

Dynamic Choices

Dynamic management involves making a decision on one asset after the agent knows of the return of another asset. The benefit comes from enlarging the set of available strategies. The increased flexibility rises the lifetime EU of the investor. The investor can duplicate the static strategy with a dynamic one, and can then adjust for knowledge of asset returns in the initial period.

The literature has heavily analyzed the effect that age has on asset allocation. While many investment professionals generally believe that younger ages can tolerate more risk, Samuelson (1962, 1989) and other academic writers believe that repeating the same investment mistakes over a lifetime does not cause risk to even out in the LT. Samuelson uses the law of large numbers to argue against such views. Gollier, 2001, notes Samuelson (1969) and Merton (1969) as showing no relationship between age and risk taking. Gollier feels this finding was hardly surprising, as HARA functions were used.

Backward induction is used to ferret out dynamic problems, whereby the 2^{nd} period problem is solved first, for each possible outcome that could exist at the start of period 2. Thus, the 2^{nd} period is contingent upon state z, obtained in the first period.

$$u(z) = \max_{\alpha_1} U(z, \alpha_1),$$

where α_1 is the optimal strategy in the 2^{nd} period. This is stated as the optimal utility, given z. One then solves the 1^{st} period problem by selecting the risk that maximizes the expectation of utility. This transforms the static situation into a dynamic problem, where an investor can adjust strategies over time. Dynamic portfolio management thus becomes a sequence of static problems designed to maximize end period utility.

The opportunity to take risk in the future raises the willingness to take risk today if the expected utility LT, v, is less concave that the expected utility ST, u. If the investor is more risk tolerant in the LT than in the ST, the optimal exposure to risk in the ST will increase in order to maximize LT utility. The agent will maximize utility LT by:

$$v(z) = \max_{c0\ldots cs-1} \sum^{S-1}_{s=0} p_s u(c_s) \quad \text{s.t.} \quad \sum^{S-1}_{s=0} \prod_s c_s = z$$

The degree of absolute risk tolerance LT is:

$$T_v(z) = - v'(z) / v''(z) = \sum^{S-1}_{s=0} \prod_s T(c^*_s)$$

The degree of absolute risk tolerance ST will be:

$$T_u(z) = - u'(z) / u''(z) = \sum^{S-1}_{s=0} \prod_s T(c^*_s)$$

Where c* is the optimal solution to v (z), above, and the above equation describes the ARA for final, LT consumption. The absolute risk tolerance of the value function is the weighted average of the degree of risk aversion of final consumption. This allows the comparison on concavity of v and u.

Gollier, 2001, notes the problem as follows: Accumulated wealth z at date 1 of active investors will be: $v(z) = \max_{\alpha} Eu(z + \alpha x\sim_1)$; the problem of young investors at date 0 is: $\max_{\alpha} Ev(w_0 + \alpha x\sim_0)$. The investor will adjust optimal risk exposure in the 2nd period on the outcome in the first. The FOC shows the relationship between optimal exposure in date 1 and wealth z, with: $Ex\sim u'(z + \alpha x\sim_1) = 0$, and optimal consumption, $c^*_s = C(\prod_s / p_s)$. We then compare $-v''(z) / v'(z)$ to $-u''(z) / u'(z)$.

If ARA is linear, a condition existing with HARA, then the concavity of u and v are the same. Thus, with HARA preferences the ability to take risk in the future has no effect on optimal exposure of risk today. LT and LT perspectives will generate the same asset allocation, and: $T_u(z) = T_v(z)$. The flexibility effect just compensates the background risk effect that goes in the opposite direction. Gollier (Optimal Portfolio Mgt, 2002) also notes that where power utility and CRRA is used, the result will still generate myopia. This was discovered by Mossin (1968), Merton (1969) and Samuelson (1969).

If Tu (z) < Tv (z), the investor should increase risk ST to maximize LT utility. If Tu (z) > Tv (z), then the investor should have a more conservative allocation ST.

Therefore, whether the investor should take more risk ST to maximize LT utility rests on the key issue of the ARA of the investor. Many finance models assume HARA, and a myopic optimality, but in reality, the ARA of the investor will be critical in determining whether adjustments of risk ST will maximize utility LT.

Allocating ST risk over a lifetime of consumption has a time-diversification effect, making agents with LT time horizons more willing to take risk. (Eeckhoudt, at 113). This effect can be stated as:

$$v(z) = \max_c \sum_{t=0}^{n-1} p_t\, u(c_t) \quad \text{s.t.} \sum_{t=0}^{n-1} c_t = z + n\,y$$

where, z is a given wealth accumulated prior to t = 0 date, p_t is the discount factor associated with date t, and z + n y is the lifetime wealth. Optimal initial risk is determined by solving for $\max_{\alpha 0} Ev\,(z\,(\alpha_0, x\sim)$. This is still the same format as the static Arrow-Debreu portfolio, except that static prices are no longer in the mix. The degree of tolerance to risk in initial wealth depends upon the sum of the absolute tolerance to risk on lifetime consumption.

If agents are patient, then they will completely smooth consumption, with all single period gains and losses allocated over the agent's lifetime. For small risks, the absolute tolerance to risk is proportional over one's lifetime. Thus, an agent expected to live twice as long as someone else will take twice as much as stock as the other person.

Gollier, 2001, also notes background risk at retirement as an additional risk upon the portfolios. This risk will have a negative effect on the demand for equities from old investors who are risk vulnerable. For example, older agents with a concave absolute risk tolerance would be in this situation for small background risk. The presence of a small risk at retirement makes the portfolio more conservative. For larger risks, this result may not hold however. And income shocks may also adversely impact and reduce exposure to risky assets (Gollier, 2001).

From Eeckhoudt, 2005: Liquidity constraints limit the time diversification effect. When adverse shocks are experienced, diversification across time will function adequately if the agent can borrow. Without liquidity reserves, negative income shocks cannot be smoothed, and the agent will become more conservative in allocation. This an argument in favor of income risks as well as DARA.

Concluding, if HARA exists, myopia is optimal with no effect on time horizons. If absolute risk tolerance is decreasing (meaning, ARA is increasing), a longer time horizon induces investors to reduce the demand for risky assets. If it risk tolerance is increasing (and ARA is decreasing), we need more information (binary, for example). Generally, a decreasing ARA function will lead to a greater demand for risk assets. Duration will increase exposure to risky assets assuming decreasing ARA and increasing risk tolerance.

When predictability of returns is brought into the analysis, LT horizon investors will take more risks early in life. Assume CRRA and a two period model. Now, not only will wealth, z, be important, but the realized return x_0 of the risky asset in the first period will factor in. If an increase in first period return results in a predictable decrease in 2nd period return (this would be an example of a predictable FSD deterioration) then the hedging demand for risk is positive (negative) if CRRA is larger (less) than unity. So, even with CRRA, return predictability will induce an agent to take more risk so long as CRRA > unity. For log utility functions (CRRA with = 1), myopia is still optimal, and the agent will not take more risk, even with predictability.

Predictability can have the same effect as DRRA when the level of optimal risk increases over time. The realized return in the 1st period, x_0, will have two different effects placed upon it. The first is a wealth effect. Because the expected return in the 2nd period will be smaller in the face of predictability, the expected final wealth will be smaller. This raises the marginal value of wealth, where v is concave in final wealth, z. The second effect is precautionary: investors will invest less in the risky asset, thereby reducing risk exposure. This reduces the marginal value of wealth. The final effect of wealth is therefore indeterminate. With CRRA > unity, the wealth effect dominates, and the hedging demand is positive. When RRA < unity, wealth effect is outweighed by the precautionary effect.

The equations involve return through each period, $x\sim_t$. Predictability exists when $x\sim_1$ is correlated to $x\sim_0$. E $x\sim_0 > 0$, due to predictability of returns. The value function is:

$$u(z, x_0) = \max_\alpha E\left[(z + \alpha\, x\sim_1 x\sim_1)^{(1-\gamma)} / ((1-\gamma) \mid x_0 \right].$$

The value function will be separable for each period, and the first period problem can now be solved via backward induction.

Gollier (Opt. Port. Mgt, 2005) states the problem as:

$$v_{n-1}(z, s_{n-1}) = \max_{c1\ldots cs} \Sigma^S_{s=1} p_s\, u(c_s) \text{ s.t. } \Sigma^S_{s=1} \prod_s (s_{n-1}) c_s = z$$

where the vector of prices in the last period depends upon states of nature s-1 that prevailed one period earlier. With CRRA > unity and mean reversion of returns, younger households should now have riskier portfolios. As stated above, this is because the wealth effect dominates the precautionary effect, with overall marginal value of wealth increasing for riskier portfolios in the presence of predictability.

When the return distribution is not perfectly known, the optimal strategy is affected. This type of parameter uncertainty will tend to make the agent more conservative than with a return structure than is completely known to be predictable. In the early stages of learning of return distributions, it would therefore be prudent of the investor to be more conservative than at a later learning stage where more understanding exists as to return predictability. This was also noted in Gollier (Opt. Port. mgt, 2005).

Merton (1969) and Samuelson (1969) were the first to develop continuous time models in the HARA context. Mossin (1969) proved that HARA functions are the only ones in which myopia is optimal in the absence of serial correlation for returns. Barberis (2000), and Campbell, Lo, and MacKinlay (1997) showed that the effect of return predictability was very large. For a 10 year horizon, 40% stock allocations are optimal without predictability, but then allocations rise to 100% with predictability.

Siegel (1998) estimated even a larger impact, when he noted that optimal equity holdings increase to over 100% (with implied shorting / margin activity) in long time frames. A time-referenced MVO model developed by the author generates similar results as to both historical periods and forward projections (Kaufhold, 2005:5). It should be noted however, that both Siegel and the MVO techniques of the author relied only on mean reversion predictability of equities in the estimation, while Barberis used a more extensive analysis.

Many other studies also indicate that stock predictability exists across time horizons. French and Fama (1988) found pronounced negative long-term serial correlations. Rozeff and Shiller (1984) found in separate articles that aggregate dividend yield may be a proxy for risk premium of stocks, with a positive relationship between dividend yields and future stock returns. Other studies have also concluded (Balvers, et al, 1990) that long-run returns on equities can be predicted so long as aggregate output can likewise be predicted.

The predictability of stock returns across time horizons is now being seen as only one of several types of predictability. Lehmann (1997) and Gollier (2002) categorized different forms of predictability, with varying impacts upon overall portfolio allocations. Predictability of long-term bonds is of a first order stochastic dominance (FSD). Predictability of a second order dominance, referred to as a means preserving spread (MPS), comes from mean reverting stocks, the stochastic volatility of returns, or learning about the size of the risk premium. Three effects of upon predictability have also identified: substitution, wealth, and precautionary. These effects may be offsetting to each other, depending upon the type of predictability and other investor level factors involved.

To Gollier (2002) in particular, the choice of initial portfolio pricing risk is driven by marginal value of wealth at the end of the initial period, which in turns depends upon the future opportunity set. If predictability raises the marginal value of terminal wealth, then it has the same effect as a decline in risk aversion. The central question becomes the extent of change in the opportunity set on marginal wealth.

With FSD shifts in return distributions, a substitution effect occurs whereby wealth increases, but it generates a smaller return, so the marginal value is smaller. However, the marginal utility of consumption increases, thus raising the marginal value of wealth. The net effect on the marginal value of wealth is negative only if the relative risk aversion is sufficiently large. Otherwise, for cases of declining risk aversion, the marginal value to an investor from a change in FSD will be positive. For MPS shifts, expected final consumption is raised, reducing the marginal value of wealth. But if the marginal utility of consumption is convex - Gollier refers to this as “prudence” - the increased risk taking increases the marginal value of wealth. The MPS reduces the marginal value of wealth only if the ratio of absolute risk aversion to absolute prudence is sufficiently large. With FSD predictability, one only needs to know whether the relative risk aversion is smaller or larger than unity to determine the hedging demand for risk. With MPS predictability, the intensity of absolute prudence versus the intensity of absolute risk aversion must be compared.

Predictability and risk aversion therefore becomes central issues. Samuelson (1994), Thorley (1995), Stangeland and Turtle (1999), and Gollier (2001, 2002) have all concluded that optimal equity allocations increase when relative risk aversion decreases. Stangeland and Turtle (1999) noted that investors experiencing reversion to the mean and negative serial correlation of assets would have decreasing risk aversion, while momentum styled investors would experience mean acceleration, increasing relative risk aversion, and positive serial correlation.

With declining relative risk aversion (DRRA) and second order RTM predictability, investors would experience a positive net utility by adopting longer time horizons and an increasing allocation to equities. If the agents have constant relative risk aversion (CRRA) whereby the proportion of stocks remains fixed as wealth is increased, then marginal utility would not be affected by time horizons. If the investor experiences increasing risk aversion so that a shift away from equities occur with wealth increases, the marginal utility actually decreases with time referenced investment policies. This is due to the agent generally being so risk averse that he or she would desire, prefer, or need the safety of the riskless asset as holding periods lengthen.

Information and Risk. Information allows for a better management of risk. In the absence of info, EU depends on a subjective probability of success. With information that conveys bad news, the probability of success goes down, leading to a decision to insure the risk. If a good signal is perceived, the probability of success can be large enough that the agent does not insure the risk.

Information is valuable because of the sensitivity of the ex post decision to the signal. In some cases, the risk should be insured regardless of which signal is received, good or bad. In such cases, the info is rather useless, as there is no change of decision on the basis of the info.

With S possible states of nature, uncertainty can be expressed as a vector of probabilities, P. The utility function is:

$$V(P) = \max_{\alpha} \Sigma^{S}_{s=1} p_s U(s, \alpha).$$

The value of info is always nonnegative, such that: $V^1 > V^0$, with V being convex. The agent can always do as well as a uniformed person if the agent ignores the info. The value of the info lies in the ability to adapt the decision. Even if the agent is risk loving, info will always be nonnegative because an informed agent not using the info will be the same as an uninformed person. The agent will then have a higher EU than an uninformed person if the info changes the decision as a resul. Thus, the choice of no info versus having info will always generate a nonnegative value to the info. In general, info allows for better management of risk.

The effect of information on behavior can be examined by the following two period model:

$$\max_{\alpha} u(\alpha_0) + \Sigma^{m}_{m=1} q^m \max_{\alpha \in B(\alpha 0)} \Sigma^{S}_{s=1} P^m_s U(s, \alpha, \alpha_0).$$

The agent selects α_0 at date 0 which generates u (α_0) in the first period. The agent then observes a signal, m, which affects his beliefs on the state of nature s. He then max EU, subject to m. Irreversibility of many investments also should factored into the process. In some instances, flexibility should be engaged in, so that the decision maker waits for new information. This would occur where the loss of future NPV as a result of the wait would be small compared with a huge irrevocable impact from immediately pursuing investment. Thus, flexibility in delaying the decision beyond date 0 may be optimal, at times. The general conclusion is that more information produces a more flexible approach to decision making in the early stages of a multi-period model, where flexibility is allowed.

The Balance of the Opportunity Set

KCK Note: Many of the following items are taken from Kaufhold, 2005:6, "Developing Investor Level Factors".

In addition to risk aversion, predictability, and utility functions chosen, many other factors may amplify or reverse the conclusions that time horizons can affect investment decisions under conditions of uncertainty. Some of the items have been briefly discussed above, including background risk, labor income risks, and liquidity constraints. All investor level factors, including predictability and risk aversion, are currently thought to comprise an investor's opportunity set that should then e optimized by a multi-variate analysis of dynamic programming. The following items summarize many investor level factors that have been identified in the literature.

<u>Effect of interest rates</u>. Campbell and Viceira (2002) at 57-58, shows that inter-temporal hedging demand depends upon PV of all future interest rates. This suggests that portfolio choices of LT investors are strongly affected by LT and persistent variations in investment opportunities than by transitory variations. The riskless asset for a LT investor will be an inflation-indexed consol bond, or a proxy for it.

<u>Discount rates</u> should decline over time (Gollier, 2003), at least with a growing economy and declining relative risk aversion preferences. This is important for macro governmental policy, as well as corporate and institutional considerations.

<u>Estimation risk</u> has been analyzed. Barberis (2000) concluded that the risk of estimation errors in forward projections should make the investor / agent more conservative in the portfolio allocations. Even considering such risk, however, Barberis believed that conservative long-term investors could still derive some utility benefit from exposure to equity allocations. If estimation is very severe however, optimal stock allocation may actually decrease over time. Thus, ignoring estimation risk could lead to too large of a position in equities.

Marginal utility of consumption (prudence) may be an important consideration in the calculation of net utility, since consumption patterns must be compared against risk aversion for second order dominance predictabilities (Gollier, 2002). Michaelides (2003) believed that in the absence of budgetary constraints, agents would generate a higher level of consumption as the consumers borrow to finance consumption. This is the case even in the presence of high return estimations. In the presence of predictability, labor income risk, and liquidity constraints however, the consumption function is found to be concave with the marginal propensity to consume quickly falling beyond a certain consumption level. The investor will have incentives to increase savings so as to increase wealth, at least when returns are expected to be relatively high. If returns are projected to be low, a decrease in savings is anticipated. Thus, to develop a greater realism in future modeling efforts, both liquidity and budgetary constraints should regularly be considered since their existence or nonexistence could make a substantial difference in allocations.

On the subject of liquidity constraints, when shorting and borrowing in the riskless asset is allowed, the investor will tend to hold a leveraged position in equities when stock returns are high, relative to the riskless rate of return (Michaelides, 2003). When shorting and borrowing is not allowed, the agent will want to hold equities to the limit when rates of return are expected to be high. Conversely, when returns are expected to be lower, in relation to the risk free rate of return, the investor will want to invest in the riskless asset, to the limit of the shorting constraint. Further, the possibility of a liquidity constraint by consumers should make the consumer less willing to bear risk presently through an increase to their risk aversion (Gollier).

Initial wealth levels have been found to be very important. Vavini and Vignola (2002) extended an overlapping generational model of Allen and Gale (1997), and concluded that initial wealth distribution of each generation was a critical factor in determining inter-generational risk sharing. Poor individuals had a stronger incentive in sharing risks across generations than rich ones. Thus, positive incentives existed for the poor to engage in diversification across time. As time horizons increased, the importance of risk sharing diminished, but the optimal risk exposure also critically depended upon initial wealth levels: the poor typically invests in risky assets for purposes of capital accumulation, while the rich prefer safe assets for capital preservation and maintenance functions. It has been shown in several studies that as wealth increases, the proportion of equity allocation also increases with many investors. This indicates not only a decreasing relative risk aversion, but also a basic shift toward capital accumulation and away from capital preservation policies with investors having DRRA preferences.

Gollier (2001) analyzes the issues of risk to wealth as:

$$v(z) = \max Eu\,(C(\pi\sim, z), \pi\sim) \text{ s.t. } E\pi\sim C(\pi\sim, z) = z$$

where v(z) is the Eu of wealth, z, who invests in risky assets; and states of the world = π. Marginal value of wealth becomes v' and absolute tolerance to risk on wealth is $- v' / v''$. Wealth, z, is now variable and no longer a constant, so we are now interested in Eu of wealth changes as total wealth, z, changes in response to more or less exposure to risky assets. Intuitively, changes in initial wealth will increase consumption levels across all states of the world, π. In the HARA case or with linear absolute risk tolerance, the Marginal propensity to consume (MPC) will be state independent, and will result in linear increases in consumption as wealth increases. In all other case of absolute risk tolerance, changes in initial wealth will generate a non-linear (ie. Convex or concave) changes in maximum consumption paths.

The age of investor and labor income risks early in life also are important to the discussion. As far back as Samuelson's early works (1963, 1969), the concern was made that labor income risks largely offset any gains made from reductions in standard deviation across time horizons. More recently, Gollier and Pratt (1996) have also found age to be relevant, with the risks of labor income acting as a substitute for portfolio risk in many instances. Younger people arguably should be more conservative in their portfolio allocations, due to the independent risk of uncertainty of labor income. Guiso and Paiella (1999) have empirically observed that that young households take on relatively less portfolio risk than more mature households. Using simultaneous modeling of both stock market predictability and non-diversifiable labor income risk, Michaelides (2003) has found that labor income risk does not seem to affect optimal portfolio choice for long horizon investors. Optimal portfolio choice is highly dependent upon the realization of factors predicting future returns, at least when predictability, liquidity constraints, and income risks are simultaneously evaluated. Gollier (2002, Opt. Port. Mgt) shows that an agent that is more adaptive in his labor supplies in response to income shocks is less risk-averse than an agent with a rigid labor supply. Gollier (2002, Optimal Portfolio Mgt) felt that there was no one right answer to allocation levels for younger agents. The answer depends upon riskiness of human capital, liquidity constraints, labor supply of the household, investment knowledge of the household, etc. Gollier concludes that considering all these factors, the share of risky assets should / may decrease with age.

As to cash in-flows into a portfolio, with CRRA and predetermined pension payments, optimality of portfolios should be independent of age. Where cash in-flows to a portfolio are variable, the agent should become more conservative on allocations as time horizons shorten. This is due to fewer periods existing in which to "time diversify" shocks. But this also provides a large advantage to agents that can make up for earlier investment / pension losses. With variable in-flows possible, the agent will save more and will become more aggressive in allocations to offset losses. In fact, Gollier (Time Diversification, 2002) feels that the younger will be less risk averse, with the ratio of absolute risk tolerance being approximately equal to the ratio of time horizons.

On a related issue to that of age and income risk, the ability of the investor to change work habits is another factor cited in the literature (Stangeland and Turtle, 1999). If a person has the flexibility to change habits and work harder so as to make up for any shortfalls that may arise, this is seen as affecting labor income risk and relative risk aversion. The labor income risks of youth may argue for a more conservative position of capital allocation at an early age, while the ability to work harder by changing work habits would argue for a more aggressive equity position earlier in life (Bodie, Merton, and W. Samuelson, 1992). Further, the possible depletion of human capital over time also factors in here, with a depletion of human capital serving to increase equities, while an increase of human capital over longer time horizons would suggest a more conservative position (Samuelson, 1994; Stangeland and Turtle).

Also, refer to the Example (below, in a following section) from Campbell and Viceira (2002) that details labor income risks in a generational dynamic model.

The ability (or relative inability) to learn of risk premiums and other investment related concepts have been noted by Gollier; and Stangeland and Turtle. RTM and predictability can eventually be taken advantage of by investors with the ability to learn over a lifetime.

The frequency of withdrawal from an investor's portfolio is a further consideration (Samuelson, 1989). The scope and timing of withdrawal patterns may obviate RTM and predictability abilities, and would thus limit an investor's risk aversion and optimal portfolio holdings.

Gusio and Paiella (1999) noted several household attitudes towards risk that might vary by demographic composition. Investor risk attitudes could dependent upon differences in occupation, portfolio composition, insurance, health related conduct, moving and job change decisions. Additionally, imperfections in the financial markets that serve as a block to investor financing of business development might make an investor more risk averse.

The existence of definite time periods of capital asset usage would make an investor more risk averse (Gollier). Two investors identically situated in terms of terminal wealth projection sand portfolio allocations would have different risk aversions if one investor has no definite future need for assets while the other has targeted time periods for capital usage. This would be consistent with many investors becoming more conservative in their investing patterns as their approach retirement age.

Gollier (2001) looks at the length of periods between trades. This length is governed by transactions costs. The effect of costs on a static model is trivial. With time periods factored in, cost may impact rebalancing and optimality. Rebalancing is contingent to the realization of the 1st period portfolio, and should induce more risk taking. Myopia will be optimal with HARA functions. (*KCK Note: But, Long-term buy and hold, DRRA investors may trade even less frequently in the face of extra costs, but then would take a higher risky allocation in* t_1 *+ periods, to make up for the costs).*

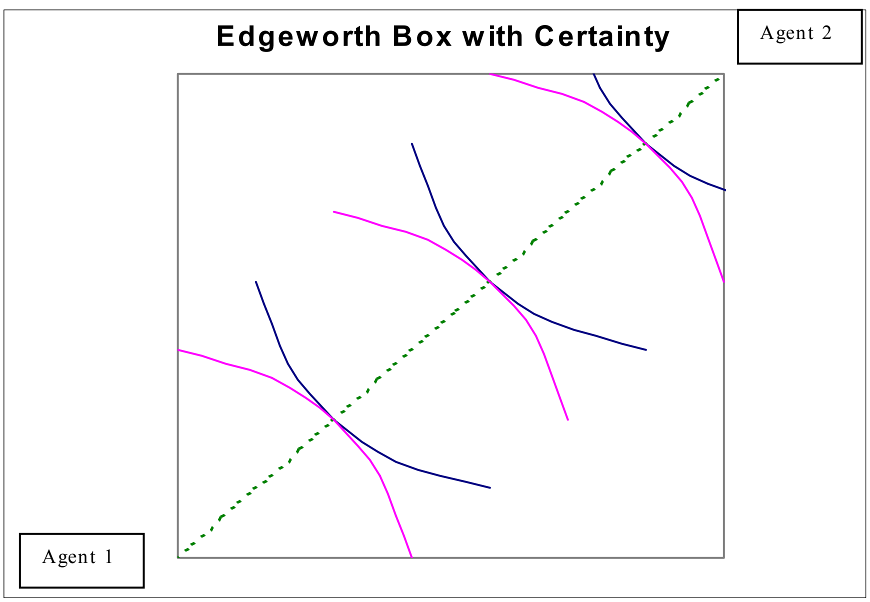
In many instances, the utility function will not be smooth, as the size of the risk taking is fixed. Gollier postulates a function with a series of discrete choices generating distinct utility curves that generally would make up a smoother and more inclusive curve, if there were enough choices with varying sizes of risk. There would be states with such a series of curves where the investor would not reject an unfair gamble, because of the non-concavity of the curves at various points. A risk averse investor may therefore engage in a risk loving state when faced with a limit on the size of the risk.

Where there is a minimum amount of risk set by regulation, the agent would take less risk in time 0 if that would result in more risk taken in time 1. This would be the case if DARA exists.

Risk Sharing and Allocation

Assume that two agents are both risk-averse and both have independent possible occurrences, one that fails and the other that succeeds. Earlier it was shown that each agent individually would diversify the risks by developing two separate choices. It can now be shown that both agents can also share risks in each of the agent's endeavors.

The static boxes previously displayed can be developed into an Edgeworth box, with the static choice of agent 2 flipped upside down compared to agent 1. The edges of the box are the quantity of the consumption choices in the two states. Where there is no macroeconomic risk, the optimal allocation that is Pareto efficient will be certainty line running diagonally between the consumption of the two agents. The optimal consumption of each agent will lie along the certainty line that is formed by the tangency of the indifference curves of both agents



With two or more agents, risk sharing or pooling is Pareto-improving for both agents. If both agents are equally risk-averse, they would equally share in the asset allocations. If one agent has more risk aversion, then it would be Pareto-optimal to transfer some of the aggregate risk to the more risk-tolerant agent in exchange for a lump sum payment. For example, if one agent has twice as high of risk aversion, then that agent should take 2/3 of the risk. The equilibrium that describes this transfer or sharing of risk is known as the contract curve, since this line is based upon mutually agreed upon risk assumption in exchange for a lump sum payment. Generally, risk allocation would occur if aggregate consumption = total wealth such that: $z(s) = \Sigma_i w_i(s)$ available in state: $\Sigma^n_{i=1} c_i = z(s)$, for all s in support of $s^\sim$.

We now assume some macro-economic uncertainty, with more to consume in one state than the other, thus giving the box a rectangular shape. Allocations with full insurance of all risks are the 45^0 lines from the origin points of the box. Because the lines do not cross, there is no possibility of full insurance for all risks. While full coverage of all risks by any individual agent will no longer be possible due to uncertainty, the risks can be shared and shifted between agents in the area of uncertainty, defined as the area between the two certainty lines of the graph, below. Assuming well-behaved markets with complete information, etc, the sharing of risks will lead to elimination of all non-diversifiable risks. This gives rise to the "mutuality principle" whereby all diversifiable risks will be washed out by mutual arrangement of agents. Thus, the general case also proves up the value of diversification.

The "contract curve" will be the Pareto-efficient allocation of risk that can raise EU of one agent without reducing EU of the other agent. The contract curve in the above graph is the dashed green line. The Pareto efficient allocation will be:

$$\text{Max}_{c(1)\ldots c(n)} \Sigma^n_{i=1} \lambda_i \, E \, u_i(c_i(s^\sim)) \text{ subject to } \Sigma^n_{i=1} c_i = z(s).$$

λ_I is a vector of possible scalars. The optimal solution can be solved state by state, as a sequence of simpler problems. The decision variables will be concave as to their functions. The probability distribution of $s^\sim$ therefore disappears in the set of efficient allocation of risk.

The non-diversifiable macroeconomic risk that was previously assumed appears as the range between the two diagonal certainty lines of the Edgeworth box. The larger the risk tolerance of an agent relative to the aggregate risk tolerance of the group, the larger the risk that particular agent will bear. The Pareto-efficient solutions will be within the diagonal lines, with agreement occurring there between agents.

If all agents have CARA preferences, then the risk tolerances will be linear in nature with the same slope (i.e. HARA). A linear sharing rule will then apply. If all agents have CRRA, all agents will share the same, constant share of the aggregate wealth. The contract curve will go from one corner of the box at agent 1 to the other corner at agent 2, and will do so in a straight line between the two diagonal certainty lines.

The planner of a group of agents, such as the CEO acting on behalf of shareholders of a firm, is known as the representative agent, since the planner is attempting to maximize EU for the entire group. The ARA of the efficient group is the sum of the individual risk tolerance in the corresponding state. With the special case of HARA and linear risk tolerance, the aggregate set of risk sharing does not matter for determining the attitude or risk by the group. For differing risk tolerances at the individual level, it will be possible to find a decision that is disagreed with by individual members of the pool.

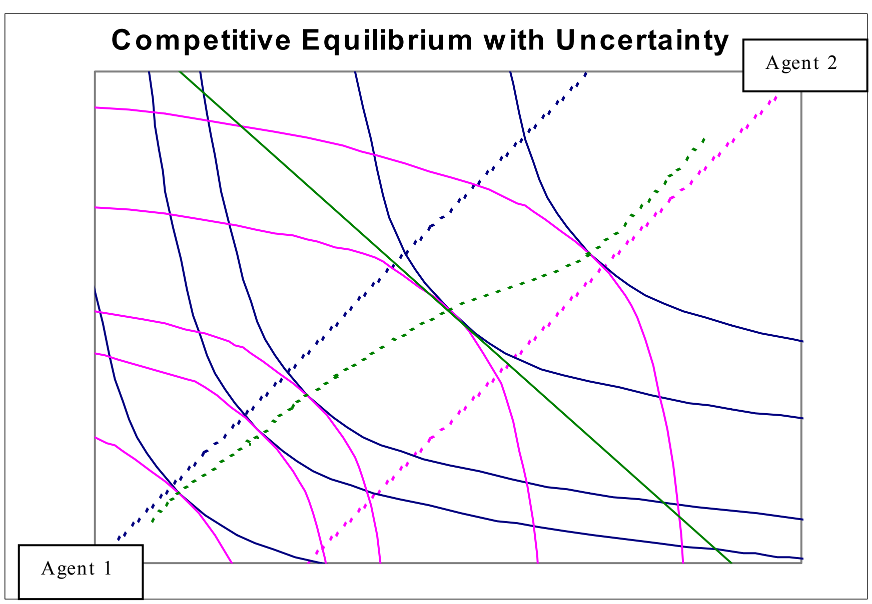
<u>Asset pricing.</u> Instead of a central planner, decentralized pricing decisions can be used to allocate risks. Assume that each agent has securities in which to exchange risk. There will also be some uncertainty about state $s^{\sim}$ occurring at the end of each period, due to macro-economic risks. Also, assume that agents can trade assets and transfer risks in a capital market. The agent *i* will find a portfolio c_i (.) of Arrow-Debreu securities that maximizes EU within a budget constraint. This is:

$$\max_{c_i(.)} Eu_i (c_i (s^{\sim})), \text{ s.t. } E\, \pi (s^{\sim})\, c_i (s^{\sim}).$$

The portfolio cannot exceed the market value of the initial endowment. A market clearing condition exists whereby aggregate consumption in state *s* cannot exceed what is available in that state. In the absence of externalities and asymmetric information, a competitive allocation will be Pareto-optimal for risk-averse EU maximizers.

With demand being a function of price, individual consumption at equilibrium depends upon the aggregate wealth available in each state. The competitive allocation will satisfy the mutuality principle, with all diversifiable risk being eliminated at equilibrium. Competitive markets will allocate the macroeconomic risk in a Pareto-efficient manner. Decentralized action produces efficient sharing of assets.

The contract curve of the Edgeworth box will be the competitive equilibrium, and a budget constraint will then define and limit the actual level of consumption by the agents. Since the agents will be risk sharing between the two certainty lines, they will have a higher EU than in autocracy. The exact position of the contract curve will depend upon the RRA of each agent, with risk tolerant agents taking more of the risk in the area of uncertainty.



The Pareto-efficient allocation can also be duplicated at the aggregate level as a single EU maximizing agent with a concave utility function. The conclusion of a market portfolio with the special case of utility has its corresponding equal in the general case with a representative agent holding the allocation that generates the market equilibrium. The risk tolerance of the representative agent, v, at aggregate wealth, z, equals the sum of individual risk tolerances in that state: $T_v(z) = \Sigma T_i(c_i(z))$.

Two fund separation theorem. With linear risk tolerances, two funds are all that is needed in a decentralized economy. One fund will offer the risky assets while the other will be the RFRR. The equilibrium condition, above, can be duplicated with just these two funds. The linear risk tolerance assumption is the only case within the EU framework where this separation theory works, with ARA being linear and with the same slope.

The RFRR and the dynamic model. The analysis can be extended to a multi-period dynamic model, instead of merely C_1 vs C_0 choices. Adding multiple time periods does not affect the properties noted above. Mutuality, macro risk, risk sharing are all still present.

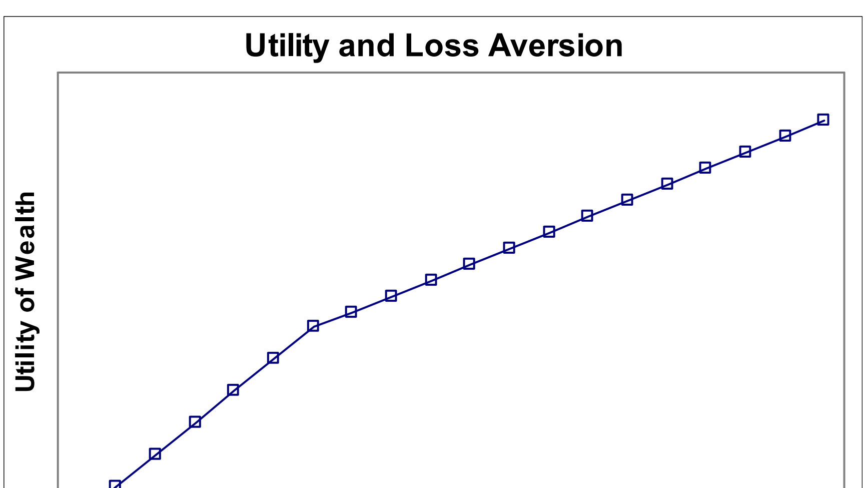
Factors affecting the RFRR in a multi-period model include the growth rate of the economy. The larger the growth rate, the higher the RFRR. There may also be zero growth, but a certain degree of probabilities of positive or negative growth in any one year. People will tend to save more with this uncertainty. This is the precautionary effect, with interest rates going down in times of recession. We do not know at any one time which of the two countervailing effects will dominate.

The yield curve can be generated by a longer time horizon in the model. With CRRA and IID of the growth rate of the economy, the yield curve will be flat.

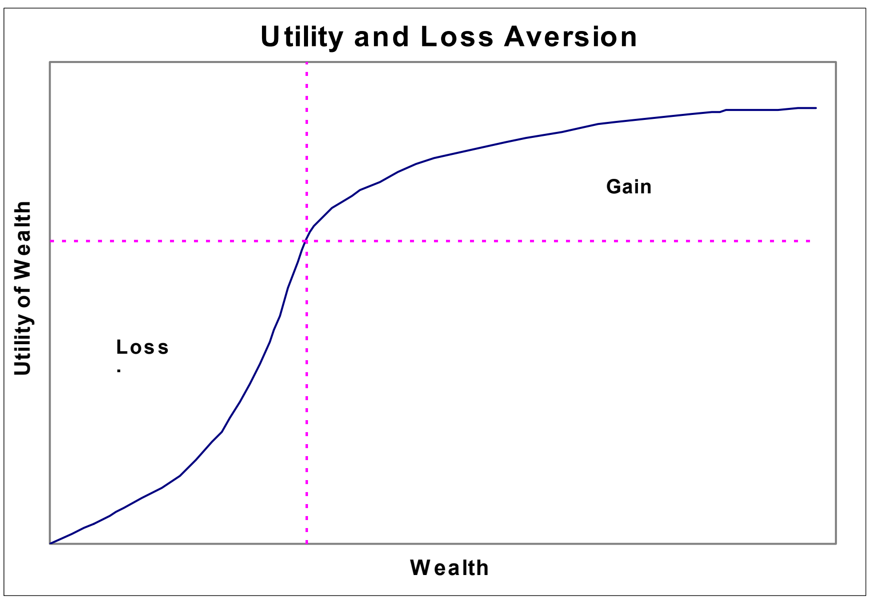
Utility and Behavior. Kahneman and Tversky (1979) believed that the utility function is discontinuous as to marginal utility, with the agent being more averse to losses than gains. This is the "loss averse" condition.

$$u(w) = aw \qquad \text{if } w <= W$$
$$u(w) = aw + b\,(w - W) \qquad \text{if } w > W$$

if a > b then the function will be globally concave in wealth. K and T felt that "a" was twice as big as "b", meaning that the agent will have the same utility with one unit of loss as two units of gain. The Arrow-Pratt approximation cannot be used in this situation, since the function is no longer differentiable. The above equations generate the following graph.



ARA will be zero everywhere except where W is undefined and infinite at the break point. Below the transition point of W, the function will exhibit IARA, and above W, the function will have DARA characteristics. K & T proposed in their 1979 article an S utility curve, as shown below.



The welfare of the agent will depend not on the wealth, but on how wealth is compared to a reference point of wealth. Each decision maker frames the economic decision with reference to some initial wealth level. Initial wealth levels have been examined in: Allen and Gale (1997); Vanini and Vignola (2002); and Guiso and Paiella (1999). Many others have commented on utility functions having behavioral aspects. See, Statman (1997); Shefrin and Statman (2000); Fisher and Statman (1999).

This type of aversion to loss may be nothing more than a particular case of the EU model. Prospect theory can be further enriched by combining it with rank dependent transformation of cumulative probabilities (RDEU), noted in a K & T paper in 1992 as cumulative prospect theory. Glaeser (2003) evaluated this model and several other theories competing with EU and concluded that behavioral objections only suggested a modification of EU rather than a refutation of it.

The Equity Premium Puzzle

(From Eeckhoudt, 2005) The equity premium is much larger than what it should be as compensation for all aggregate risks being absorbed into the representative equity portfolio. There have been large amounts of research to understand this, including labor income, wealth distribution, liquidity constraints, etc. There may be no single explanation for the equity premium, however. *[KCK note: a time horizons view actually worsens the puzzle, since pricing risk completely disappears in the longer holding periods]*.

(The following is from Gollier, 2001) Solving for the first order condition of the generalized version of the standard portfolio problem, $\max_{\alpha i \ldots \alpha n} \text{Eu } \Sigma^{n}_{j=1} (\alpha_I \, y\sim_I + (1-\alpha_I) P_j)$, by setting the equation = 0 and taking the first derivative, yields a "pricing kernel" that establishes price p for asset i. Commonly, the property is expressed in stock returns rather than pricing or net cash flows. The expected return will be the rfrr + a risk premium, with the premium being smaller for a larger covariance with the pricing kernel. The covariances will be large when marginal utility is large, with low consumption, making the asset return high. The equilibrium price of the macro risk = expected return of stocks compared with return of a safe strategy (ie. the rfrr). This is the equity premium. An increase in risk aversion will increase the supply of stocks, reducing the price, p. This increases the overall return, thereby increasing the equity premium. This makes sense, since we need higher returns to induce an investor with greater risk aversion to bear the risk.

Gollier notes that the theoretically derived equity premium is only 0.1% higher than the return of bonds. The observed equity premium in the 20th century has been 6%. These results are consistent with Lucas (1978) and Prescott (1985).

Gollier believes that the equity premium is 20 times more than predicted by the Eu model due to a series of simplifying assumptions, including no risk to human capital, lack of international al activity in the model, consumption of all wealth at the end of period 1 (i.e. a single period model), no trading costs, no options or alternatives are considered, CRRA and identical investor preferences being assumed, and access exists to the financial markets by all agents.

Gollier suggests that the overly large observed premium may be the result of historical data having catastrophic losses, while the market realizes this possibility will eventually come to pass and prices the assets accordingly. Also noted is the argument that not all agents are equity participants, for various reasons. This would increase the supply of equities (or decrease demand?), therefore reducing the equilibrium price. The premium would increase.

Also, the model has not brought in international markets. Increasing cash flows between financial markets would have the effect of raising aggregate demand, and thus market pricing. This increases equilibrium pricing and reduces the equity premium.

Gollier, 2001, at 131-132 notes that adding uninsurable risk to human capital may help to solve the equity premium puzzle, but Gollier feels that the noise would have to be on the order of 80% standard deviation of the annual growth rate in labor income to explain the premium puzzle. In isolation of other risks, background noise cannot explain the equity premium puzzle.

Gollier, 2001, at 173: With dynamic portfolio management, older investors may become more risk averse (concave AR tolerance; convex ARA). This makes them more risk averse than in the one period model. This will increase the risk premium needed to induce the taking of risk.

Examples and Comparisons

An example of long horizon portfolio modeling. The following summarizes important findings of an involved life-cycle model developed in Campbell and Viceira (2002), at 195-225. Since this study is an excellent example of a dynamic model using several utility factors, including risk aversion, predictability, lognormal utility functions, and labor income risks, it will be extensively reviewed an analyzed.

Allocations over an investor's entire life can be modeled using many of the above concepts. With a power utility function, an agent's multi-period consumption can be mathematically described as:

$$C_{kt}^{(1-\gamma)} / (1-\gamma) = E_t \Sigma_{I=1}^{T-1} \delta^i (\Pi_{j=0}^{i-1} p_{t+j}) (C_{k,t+1}^{(1-\gamma)} / (1-\gamma))$$

where, C is the level of consumption on date t; γ is the coefficient of relative risk aversion which is assumed to be > 0; the discount factor is δ, and is assumed to be < 1; p_t = probability that investor is alive at date t+1. Work life is assumed to be from age 20 for high school graduates, and 22 for college graduates, until a typical retirement age of 65. Expected life probabilities are used for life beyond retirement, with a further assumption of 1.00 probability of death by age 100. Other assumptions and conditions also apply.

In a dynamic setting, the optimal portfolio problem is: max Eu is a $\int U(C, t)dt$, s.t. continuous time inter-temporal budget constraint. Consumption toady is achieved at the cost of consumption tomorrow, and there is a balance between them. Optimality is a function of the risk premium, inversely proportional to the volatility and relative risk aversion, and then the hedging component. This is stated as a bellman equation:

$$Uc = J w$$

$$A\sim = 1 / (-J_{ww} W/J_w) * ((u_p - r)/var_p) - (J_{ws} / J_{ww} W) * (\text{std dev}_s / \text{std dev}_p)\ rho_{ps}$$

An approximate analytical solution can then be attempted. The investor will attempt to maximize the above equation, subject to several work-life and retirement budget constraints. The solution is obtained via backward induction, as was done in the section on dynamic modeling.

Using empirical data obtained from the Federal Reserve Board's Survey of Consumer Finances and other sources, Campbell and Viceira (2002) demonstrates that consumption is linked closely to net income, with little savings occurring outside of retirement accounts until after age 40. Consumption rises with income early in life because of borrowing constraints, and then falls later in life. Consumption paths are not completely smoothed, with an increase early in life, followed by a decrease in the later years. Such a pattern of incomplete smoothing is consistent with the above discussion on inter-period consumption activities. Total wealth and financial wealth is distinctly humped shaped, with a peaking occurring in the 1st year of retirement, and then sharply trailing off from there.

Campbell and Viceira also have extensive comments regarding human wealth (Id, ay 162-169). Human wealth can be considered a from of investment, being the PV of the future labor earnings. This form of wealth is non-tradable. RFRR has a constant log return rf per period, or rf ≡ log(1 + Rf). Stocks have log returns of rt+1 / period, and constant excess log return of $E_t\ (r_{t+1} - rf) \equiv \mu$, and variance $\sigma^2 \equiv Vt\ (rt+1)$. (at 162 et seq).

$$\text{Total wealth} = Wt + Ht,$$

where Wt is the financial wealth and Ht = labor wealth. Unconstrained,

$$\alpha\sim = (\mu + \sigma^2/2)/\ (\gamma\ \sigma^2)$$

where γ is RRA. The risky asset, stocks will be invested as: α~ (wt + Ht), and the riskless asset (assuming two assets, one risky and one riskless) would be: (1- α~) *(Wt + Ht). Given the constraint in human wealth, how much should he allocate Wt to stocks and the riskless asset? Assuming Ht is riskless (or at least very stable in income), then the matter is easy. The implicit holdings in Ht becomes the riskless asset, and thus Wt should be adjusted to so that total wealth has the unconstrained levels of the riskless asset. To this extent, Ht just substitutes for the RFRR. Stocks will have: α~(wt + Ht), and the riskless asset will be (1- α~) *(Wt + Ht) – Ht. Optimality will be:

$$\alpha = \alpha\sim(Wt + Ht) / Wt = (\mu + \sigma^2/2)/\ (\gamma\ \sigma^2)\ (1 + Ht / Wt).$$

α will now be > α~ since Ht and wt are nonnegative.

At retirement, Ht / Wt = 0, since there will be no labor income. Early in work life, the ratio Ht / Wt will be very large, for two reasons: 1) the agent will have large amounts of PV of future labor income; and 2) but little or no financial wealth. Over the work life, the Ht remaining will decrease will wt will typically increase. This lowers the ratio over time until it is zero at retirement. Because this ratio is in optimal allocation equation, the share of total wealth devoted to risky assets will decrease over time.

This shows that an agent with riskless, nontradable labor income (ie stable, idiosyncratic income having no likelihood of variability) should weight the financial portfolio towards stocks more so than an agent who only owns tradable assets, and has no labor income streams. This is an important observation, since it tends to explains why younger people with stable employment will tend to be heavily concentrated in equities, while older people at retirement and with no labor income at all will be more conservative in risky allocations.

In reality, labor earnings are uncertain for many agents, making the human investment risky. This As the variance of labor income increases, the tilt towards risky assets decrease. At the limit, the allocation of risky assets approaches that of a retired agent with no labor income. Starting with rf ≡ log(1 + Rf); and with the risky asset being a constant expected log excess return Et(rt+1 – rf) ≡ u. Then:

$$\max E_t (\delta (C_{t=1}{}^{1-\gamma} / (1-\gamma))) \text{ s.t. } C_{t+1} = W_t (1+R_{p,\,t+1}) + L_{t+1}, \text{ with } R_{p,\,t+1} = \alpha_t (R_{t+1} - Rf) + Rf.$$

The optimal allocation to the risky asset has two components, the first describing the uncorrelated labor income risk. The 2nd component is an income hedging item, where the risky asset desirability depends upon the ability to hedge against a bad realization of labor income. Optimality of the financial risky asset will now be:

$$\alpha_t = (1 / p) ((u + \sigma_u^2 / 2) / (\gamma \sigma_u^2)) + (1 - 1 / p) (\sigma_{Lu} / \sigma_u^2)$$

where the wealth elasticity of consumption is "p". The first component on the left side of eth + sign is the labor income risk, and the right side of the + side is the hedging component. If the covariance between the risky asset and the labor income is negative, the risky asset offers a good hedge against income shocks, and would increase the optimal allocation. Where labor income is idiosyncratic ($\sigma_{Lu} = 0$), the 2nd component drops out, and the introductory material then applies, with Ht substituting for the RFRR. Risky allocations of the financial assets would thereby increase.

The text analyzes the question of variability of labor income in terms of background risk. The conclusions of the analysis regarding background risk apply here, too. The increase in variance of labor income is assumed to be a mean-preserving increase. We also assume that labor income is uncorrelated to equities. Optimality becomes:

$$\alpha_t = (1 / (p (l, r_p))^* ((u + \sigma_u^2 / 2) / (\gamma \sigma_u^2))$$

Sufficiently conservative agents with risk aversion > 1/ p should reduce their financial exposure to risky assets in the presence of labor income variance. If labor income is correlated with risky returns, optimal risky allocations decrease. Where there is a completely positive correlation between labor income and risky assets (massive ownership of an employer's stock, for example), the agent should tilt away from the risky asset, since human investment becomes an implicit investment in that risky asset.

The crucial variables for purposes of asset allocation are liquid wealth, retirement wealth and future labor income. Labor income is observed to act as a substitute for the RFRR, with only a weak correlation between labor income and risky financial assets. Early in life, the investor will want to invest fully in equities, to the extent that he or she can. Borrowing constraints and a lack of liquid reserves will limit broader access to the financial markets at this stage. From age 40 onward, liquid wealth increases relative to future labor income, so that a shift in allocations gradually occurs away from risky financial assets.

Agents with greater precautionary tendencies will consume less and save more until retirement. Once at retirement, these investors will then consume more, since there is no longer any labor income risk to contend with, and retirement wealth may also be in a riskless annuity. Highly risk averse investors may have a lower portfolio allocation devoted to equities. Interestingly, the model predicts that such risk averse agents may still increase their equity percentage as they age, but will start from lower initial level of equities. For impatient households, consumption is greater early in life, and less later on. They accumulate almost no wealth by the normal age of retirement.

Risky assets should be very attractive to young households with many years to retirement, since human wealth will act as a substitute for the RFRR, allowing all financial assets to be moved into risky assets. The attractiveness of risky assets is reduced later in life as human wealth declines and financial assets accumulate. This is consistent with long-term perspectives on the risk-free asset, with the long bond or an inflation-indexed bond representing the risk-free instrument. Thus, investors at or near retirement will reduce risky asset allocations, and move to real bonds in an attempt to match near-term liabilities.

In summary, riskless labor income creates a strong portfolio tilt toward risky financial assets. Variability in labor income can reduce that tilt, but not reverse it. Only if labor income shocks are highly variable and strongly positive in correlation to risky financial assets will an investor with labor income hold a more conservative portfolio than a retired investor with no labor income. Normally, labor income risk will only serve to generate a conservative portfolio at the limit that simulates a retiree's portfolio.

<u>Comparison with Investment Behavior and Finance Practitioner Advice.</u> Comparing general utility concepts with observed investor behavior and financial planning advice is very useful in ascertaining whether the multitude of financial equations adequately describes the investor's economic choices under conditions of uncertainty.

With power utility, DARA assumptions and return predictability, dynamic management suggests that long-term investors should have a high percentage of wealth in risky investments. This is generally consistent with professional investment advice as well as actual behavior of many long-term investors who keep the bulk of their assets in either private businesses or publicly-traded instruments.

The theoretical conclusions of dynamic efforts actually produce a much higher equity percentage than what has generally been used by long-horizon investors, however. As is shown in the above study and example of Campbell and Viceira (2002), leveraging and shorting activities on equities would be optimal, at least before tax impacts are considered and with no liquidity constraints assumed. Almost no individual investors with a long-term leaning engage in such activities, and many institutionals are even reluctant to leverage or short to the extent of the modeling recommendations. Even when liquidity constraints are imposed upon the financial models, equities should comprise close to, or be at, 100% of the allocation, in market environments where risky assets are performing better than the RFRR. This is still far too aggressive for many, if not, most long-term investors. In fact, many households own few if any risky assets.

Additionally, many younger investors will start out their investing careers rather conservatively, building up to significant equity percentages only after much time has elapsed. This may partly be explained by the ability to learn as well as a possible shift in work habits, but the fact that long horizons investors, many of whom will be young, do not invest nearly as aggressively as suggested by some dynamic models implies that labor income risks and / or the lack of investment acumen and knowledge in the early years may be more significant than modeled.

Additional utility factors only serve to aggravate an all-equity allocation. Stable labor income (as well as retirement revenues via Social Security payments), the ability to learn of investments, and the further ability to change work habits all drive consumption based models towards very high equity allocations. It is only when labor income is assumed to be highly variable and / or positively correlated with risky asset variability that dynamic modeling efforts would reduce equity allocations. Even then, the allocations would only approach that of a retired investor. The addition of background risk, estimation risk and parameter uncertainty, the existence of definite time periods for withdrawal, and the existence of a defined frequency of withdrawal may also serve to lower the risky asset recommendation. When the full opportunity set is considered, it is possible although certainly not guaranteed, that equity allocations would become more realistic and in accord with actual investor behavior.

Some studies empirically show that labor income and its variability is heterogeneous. This can be modeled rather well with general utility processes, although the results become industry-specific in many instances.

While many of the theoretical efforts overemphasize equity allocations for long-term investors, dynamic modeling may adequately predict and explain the behavior of older investors who pull back on equity allocations as they approach retirement. With inflation-indexed bonds (or long bonds in a low inflationary environment) being considered the RFRR in real terms, bonds take on an increasing important role in capital preservation efforts as an investor's time horizon shrinks. Such bonds make an ideal match for short-term cash flow needs. Conclusions reached with the dynamic models are thus generally consistent with investor behavior and financial advice for retired individuals.

The desire to smooth consumption over the course of one's life is a hallmark of the lifestyle consumptions models. Empirical tests have shown that investors are not able to completely smooth consumption. This lack of complete smoothing can be theoretically modeled by the introduction of labor income shocks, borrowing and liquidity constraints. For example, when an investor suddenly loses a job and has few if any opportunities to secure loans, the models could generate a drop in investor consumption.

Diversification is theoretically optimal with either the special or general case of utility, so much so that the default recommendation is for broad diversification of assets. The preference of many long-term investors for a relatively concentrated and focused portfolio is at least supported by studies showing almost all non-systemic risk to be eliminated with as few as 50 to 75 assets in the portfolio.

The Edgeworth Box is superbly suited for the modeling of risk sharing among investors. By envisioning a decentralized economic process at the individual level, the allocation decision can be clearly seen for what it is: the shifting of systematic, macro-level risks among willing investors. Highly risk averse investors will be willing to pay others to take market-level risks, while risk tolerant investors will be willing to assume these risks in exchange for some form of compensation, typically via the risk premium.

The fact that dynamic modeling does a good job, but far from a perfect one, at predicting actual investor behavior merely shows the need for further research on the subject. With either a timeless horizon assumption as per current portfolio theory or a time-variable policy as with dynamic programming, economic riddles still exist. The answers may very well lie in continued research on the investor opportunity and the equity premium puzzle. It looks as though no one, single variable or concept will solve the problem. Answers may come only from a more involved model with interdependent and simultaneously changing variables. The overall theme of maximizing consumption over a lifetime is at least a viable and capable one for investors with varying time horizons.

References

F. Allen and D. Gale, "Financial Markets, Intermediaries and Intertemporal Smoothing, *Journal of Political Economy* 105 (1997): 523-546.

Kenneth Arrow, "Liquidity Preference," Lecture VI in *Lecture Notes fpr Economics 285, The Economics of Uncertainty*, at 35-53, Stanford University (circa 1963);

Ronald Balvers, Thomas Cosimano, and Boll McDonald, "Predicting Stock Returns in an Efficient Market," *Journal of Finance* 45, no. 4 (September 1990): 1109-1128.

Nicholas Barberis, "Investing in the Long Run When Returns are Predictable", *The Journal of Finance,* 2000, no. 1, pp. 225-265.

Richard Bellman, *An Introduction to the Theory of Dynamic Programming,* Rand Corporation (1953).

Daniel Bernoulli, "Exposition of a new theory on the measurement of risk." (1738).

Bodie, Merton, and W. Samuelson, "Labor Supply Flexibility and Portfolio Choice in a Life Cycle Model", *Journal of Economic Dynamics and Control* 16, 427-49 (1992). 1992.

Arthur Lyon Bowley, *The Mathematical Groundwork of Economics* (1924).

John Y. Campbell and Luis M. Viceira, *Strategic Asset Allocation,* Oxford University Press, NY (2002; reprinted 2003).

John Y. Campbell, Andrew W. Lo, and A. Craig MacKinlay, *The Econometrics of Financial Markets,* 1997.

Louis Eeckhoudt, Christian Gollier, and Harris Schlesinger, *Economic and Financial Decisions Under Risk,* Princeton University Press, 2005.

Francis Edgeworth, *Mathematical Psychics,* 1881.

Edwin J. Elton, Martin J. Gruber, Stephen J. Brown, and William N. Goetzmann, *Modern Portfolio Theory and Investment Analysis,* John Wiley & Sons, 6th ed., 2003.

Eugene F. Fama and Kenneth R. French, "Dividend Yields and Expected Stock Returns," *Journal of Financial Economics* 22 (Oct. 1988): 3-25.

Kenneth L. Fisher and Meir Statman, "A Behavioral Framework for Time Diversification", *Financial Analyst Journal* (May / June, 1999): 88-97

E. Glaeser, "Psychology and the Market", NBER Working Paper 10 203 (2003).

Christian Gollier -----

- "Time Horizons and the Discount Rate," 2003.

- "Optimal Dynamic Portfolio Risk with First-Order and Second-Order Predictability," (2002).

- "Time Diversification, Liquidity Constraints, and Decreasing Aversion to Risk on Wealth", Christian Gollier, *Journal of Monetary Economics,* Vol. 49, No. 7, October 2002, p. 1493-1459.

- Economics of Risk and Time, MIT Press, 2001.

- With John Pratt, "Risk Vulnerability and the Tempering Effect of Background Risk", *Econmetrica* 64, 1109-23 (1996).

Luigi Guiso and Monica Paiella, "Risk Aversion, Wealth and Background Risk," first draft 12-6-1999.

Daniel J. Kahneman and A. Tversky -----

- "Prospect Theory: An Analysis of Decision under Risk", Vol. 47, No.2 (1979): 263-291.

- "Advances in Prospect Theory: Cumulative Representation of Uncertainty", *Journal of Risk and Uncertainty* 5 (1992): 297-323.

Kevin C. Kaufhold -----

- "A Philosophy of the Long-Term", draft book (circa 2007).

- "Developing Time Horizons for Use in Portfolio Analysis", IFEBP Benefit and Compensation Digest, March, 2007, WebExclusive.

- "Applying Time Horizons to Portfolio Analysis", 2006:Summary.

- "Developing Investor Level Factors for Use in Portfolio Analysis," 2005:6.

Mark Kritzman and Don Rich, "Beware of Dogma," *Journal of Portfolio Management,* Summer, 1998, p. 66-76.

Lehmann (1997).

Robert E. Lucas, Jr. "Asset Pricing in an Exchange Economy", Econometrica 46, 1429-46 (1978).

"Lifetime Portfolio Selection Under Uncertainty: The Continuous Time Case", Robert Merton, *Review of Economics and Statistics,* vol. 51, no. 3 (1969), pp.247-257.

Alexander Michaelides, "Portfolio Choice, Liquidity Constraints and Stock Market Mean Reversion", March 14, 2003.

Jan Mossin, "Optimal Multiperiod Portoflio Policies", *Journal of Business* 41, 205-25 (1968).

Vilfredo Pareto, *Manual of Political Economy* (1906).

J. Pratt, “Risk Aversion in the Small and the Large,” *Econometrica,* 32 (1964): 122-136.

Prescott (1985).

Stephen Ross, “Adding Risks: Samuelson’s Fallacy of Large Numbers Revisited,” *The Journal of Financial and Quantitative Analysis* 34, no. 3 (Sept 1999): 12-21.

Michael Rozeff, “Dividend Yields on Expected Risk Premiums,” *Journal of Portfolio Management* 11, no. 1 (Fall 1984): 68-75.

Robert Shiller, “Stock Prices and Social Dynamics”, *Brookings Papers on Economic Activity* 2 (1984): 457-510.

Paul Samuelson -----

- “The Long-Term Case For Equities And How It Can Be Oversold”, *Journal of Portfolio Management,* vol. 21, no. 1 (Fall 1994): 15-24.

- “The Judgment of Economic Science on Rational Portfolio Management: Timing and Long-Horizon Effects”, *Journal of Portfolio Management*, 16 (1989), 4-12.

- Paul Samuelson, “Lifetime Portfolio Selection by Dynamic Stochastic Programming”, *Review of Economic and Statistics,* vol. 51, no. 3 (August 1969): 247-257.

- Paul Samuelson, “Risk and Uncertainty: A Fallacy of Large Numbers”, *Scientia,* vol. 57, no. 6 (April / May 1963): 1-6.

Jeremy .J. Siegel, *Stocks for the Long Run,* McGraw-Hill, 1998.

H. Shefrin and Meir Statman, “Behavioral Portfolio Theory”, *Journal of Financial and Quantitative Analysis,* 2000, no. 2, pp.127-151;

David Stangeland and Harry J. Turtle, “Time Diversification: Fact or Fallacy”, *Journal of Financial Education,* Fall 1999, at 1-13.

Meir Statman, “Behavioral Finance,” *Contemporary Finance Digest* 1 (Winter, 1997): 5-22;

Steven K. Thorley and Craig Merrill, “Time Diversification: “Perspectives from Option Pricing Theory”, *Financial Analysts Journal*, May / June 1996.

Paolo Vanini and Luigi Vignola, “Optimal Decision-Making with Time Diversification”, *European Finance Review* 6: 1-30, 2002.

von Neuman and O. Morgenstern, *Theory of Games and Economic Behavior, J.*, Princeton University Press, 1948.