Inter-Temporal Investment Models

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Working Paper 2008:3
October, 2008

Introduction

The inter-temporal modeling of assets and liabilities has a long and rich history. Many early efforts concluded that time played little or no role in investment decisions. More recent papers and books discuss the issue from a general utility perspective, generally believing that allocations will vary according to any number of factors in an investor's opportunity set, including the intended holding period.

In this paper, a representative sampling of literature is noted and reviewed in the appendix. This literature review was used for numerous sections of Kaufhold, draft book, *A Philosophy of the Long-Term.* Those sections were then structured into one document and presented below.

Discussion on the Relevancy of Time

Initial Efforts. Research has been conducted over the years on the impact to the CAPM from a relaxation of its assumptions. By the 1960's, significant research was occurring on relaxing the unitary time horizon assumption. Discrete and continuous time models evaluated whether longer time horizons would affect allocation decisions. Inter-temporal models attempted to integrate time horizons into the theoretical constructs of the emerging portfolio and asset pricing models.

Paul Samuelson was one of the first to write on the topic of time horizons.[1] Samuelson showed that investor utility does not increase in longer holding periods merely because of a reduction in the standard deviation of returns. Using a discrete time model, Samuelson assumed constant relative risk aversion (CRRA) and log utility. He supplied proof that for an isoelastic marginal utility function, $U'(C) = C^{\gamma-1}$, $\gamma < 1$, the optimal portfolio decision is independent of all consumption savings decisions. This leads to a constant percentage allocation. Samuelson also argued that a fallacious interpretation of the law of large numbers typically led to the recommendation that risky allocations should increase over time. [2]

Merton developed an early version of a continuous time model, supporting Samuelson with similar conclusions.[3] Merton uses a Bellman dynamic equation to derive optimality. Merton also assumed CRRA as well as log utility, where:

$$U(C) = \log C, \text{ and } U(C) = = C^{\gamma} / \gamma, \\ \gamma < 1, \text{ and } \gamma = 0.$$

The portfolio problem was generally stated as:

$$\text{Max}_{\{C,\,w\}}\ \Phi\ (w, C; W; t) = 0$$

where, w is the proportion invested in the risky asset; C is consumption; W is total wealth; and t is time. Merton concluded that the optimal portfolio rule for an infinite horizon model with CRRA, the allocation decision is independent of the consumption decision. “The price change and the resulting level of wealth have zero relevance for the portfolio decision and hence it is constant”. [4]

A time invariant result occurs with log utility even where asset returns are serially correlated. The technical reasoning for a time invariant result is direct: An investor will maximize expected log returns:

$$r_{pK,\,t+K} = r_{p,\,t+1} + \ldots + r_{p,\,t+K}$$

Long-horizon K-period log returns will merely be the sum of one-period log returns. This sum will be maximized by choosing in each period the allocation that is optimal for a single-period log investor.[5]

Merton also looked at CARA utility where: $U(C) = -\,e^{-\eta C} / \eta$, $\eta > 0$. With this function, consumption was no longer a constant proportion of wealth (as is the case with log utility), but it was still linear with wealth. Instead of the proportion of wealth invested in the risky asset being constant, the total dollar value of wealth invested in the risky asset was constant. In such a case, the total value invested in equities should remain the same across time.

Merton’s works were so significant that they still serve as a primary reference point to this day. The impact of these early efforts by Samuelson and Merton were so great that they have even been described as “mathematical truths”.[6] Samuelson has more recently used behavioral concepts to argue that a reduction in risk in long holding periods is illusionary because cognitive errors result from the belief that losses can never occur in long holding periods.[7] As recently as 2003, a standard portfolio text considered time diversification to be a fallacy due to the increasing uncertainty of the ultimate portfolio value over time.[8]

Two other great economists, Eugene Fama and Merton Miller weighed into the discussion with multi-period consumption investment models. [9] A discrete time multi-period model with dynamic programming was used. The dynamic process was to maximize expected lifetime consumption, $U_t\,(C_{t-1}, w_t \mid \Phi_t)$:

$$U_t\,(C_{t-1}, w_t \mid \Phi_t) = \max_{ct,\,h} \int_{\Phi t+1} U_{t+1}\,(C_t, \textstyle\sum^{N}_{j=1} h_f\,[1 + Rj\,(\Phi_{t+1}\,])\,\Phi_{t+1}\,dF_{\Phi\,t}\,(\Phi_{t+1})$$
$$\text{subject to: } 0 <= C_t <= w_t \text{ ; and } \textstyle\sum^{N}_{j=1} h_f = w_t - c_t$$

where c = consumption in time t; w is wealth in time t; and Φ is the "state of the world". The solution uses a sequence of two-period problems that are solved to maximize lifetime consumption. The authors provided a theorem that risk averter investors having a strictly concave function of lifetime consumption will have derived utility functions in each time period possessing the same properties as the long-term solution.

These early efforts at modeling assets over time gave theoretical support to the unitary time horizon assumption of portfolio theory. The studies themselves were generally limited to HARA utility, log functions, or power equations with CRRA and IID assumptions. Even though the studies involved a limited set of utility functions and assumptions, the academic community largely accepted Samuelson's premise of time irrelevance.[10] This was at severe odds with financial practitioners, who regularly and routinely advised clients to increase equity allocations in longer holding periods.

As a result of the initial papers on inter-temporal modeling of assets, the prevailing intellectual view at the end of the 1960's was that time horizons did not matter in investment policies. In 1973, Robert Merton provided far greater sophistication to the analysis, effectively moving the discussion towards a time-oriented view.[11] Merton's model replaces dependence on quadratic utility with continuous time equations that adds other factors to the analysis, becoming a multi-beta process. Merton used an Ito process for returns rather than quadratic utility. Since only the first two moments of a probability distribution matter with this method, the results could still be stated within means-variance space.

With constant risk tolerance utility functions and investment opportunities, optimal portfolio choices are also constant. When the investment opportunity set changes, so will portfolio allocations. The decision rule as to optimal allocation thus becomes dependent upon risk aversion and other factors of an opportunity set, rather than being fixed as was the case with log utility or HARA equations.

Believing that current demands are affected by the possibility of uncertain changes in future investment opportunities, Merton showed that changes in interest rates produces shifts in the investor's opportunity set. Merton's theorem was that all risk-averse investors will be indifferent between portfolios composed of three funds, namely the two funds traditionally associated with the mean-variance efficient frontier, plus a third one to hedge against unfavorable inter-temporal shifts in the frontier. At equilibrium, investors will be compensated for bearing systematic risk and for bearing risks of unfavorable shifts in the opportunity set. The optimal portfolio chosen at the initial date will not only be on the efficient frontier, it must also provide the best hedge against future changes in the opportunity set. A single-state ICAPM can be written as:

$$\alpha_i = r_f + \beta^m{}_i\,(\alpha_m - r_f) + B^h{}_i\,(\alpha_h - r_f)$$

where $\beta^m{}_i\,(\alpha_m - r_f)$ is the market risk premium and $B^h{}_i\,(\alpha_h - r_f)$ is the hedging component for uncertain future events affecting the opportunity set. An econometric statement useful for regression analysis would then be:

$$\alpha_i - r_f = \beta_0 + \beta^m{}_i (\alpha_m - r_f) + B^h{}_i (\alpha_h - r_f) + \varepsilon_i$$

Each additional state of a multi-beta model would have a term like $B^h{}_i (\alpha_h - r_f)$ for each state variable. With such a system, the standard CAPM effectively becomes a special case of the ICAPM, limited to situations where investors do not or cannot hedge against future adverse events and where no additional state variables come into play. The hedging component would drop out as well as all other possible state variables, leaving only the CAPM's single-factor equation.

Time is far from irrelevant with such a process. With changes in future investment opportunities affecting current asset demand, investment policies should identify the relevant holding period in order to appropriately hedge against adverse inter-temporal changes.

After Merton's seminal works, research on the topic became ever more involved, expanding considerably in many directions. Consumption based CAPM models, more extensive expected utility theories, and dynamic programming methods were all developed. Most of these models use time as an integral component. The more recent models can be reviewed via a discussion of input assumptions, as well as through a discussion of the opportunity set. Merely changing the classical assumptions, as discussed below, has a large impact on time-related investment concepts.

Relaxing the Assumptions. Using the classical assumptions of HARA utility or IID, CRRA, and power utility will generate constant utility across time horizons.

The use of CRRA largely stems from convenience rather than descriptive accuracy.[12] It is quite possible that different investors, or the same investors at different points in life, will have differing preferences on risk aversion. Indeed, the lack of consensus on what type of relative risk investors possess is indicative of shifting and differing risk aversion preferences among investors. [13] A CRRA assumption has been defended, by noting that aggregate financial variables must be stable in the face of secular economic growth.[14] This does not negate the possibility that investors at the individual level will vary their relative risk aversion tendencies over their lives. It is also quite possible that different investors will have different risk aversions in response to the same economic stimuli at the same moment in time.

As to the return distribution assumption, an independent and identical return distribution has been described as being "highly restrictive".[15] The discussion of return predictability has clearly shown that over longer time horizons, pricing returns are not independent. Instead, returns are mean accelerating in the short run, yet highly mean reverting in the long run.

Regarding the selection of an appropriate utility function, log and HARA types of equations exhibit the same utility regardless of the length of the holding period. This was the precisely the result of early multi-period models in the 1960's.[16] By using CRRA assumptions and HARA utility functions, continuous time models found no relationship between an investor's age and risk taking. The finding is largely attributable to the choice of the utility function employed, not an overall conclusion on the relevancy of time horizons. [17]

Any equation that generates increasing absolute risk aversion, as is the case with the quadratic, arguably should not be used in the analysis since IARA is inconsistent with the consensus on investor behavior. Optimized asset allocations across time will likely be inconsistent with investor risk preferences, since increasing absolute risk is modeled with the quadratic instead of decreasing absolute risk.

The continued use of the quadratic has been defended however, by the realization that stochastic dominance procedures will still lead to the same portfolio allocations as with MVO. [18] Such a realization is quite useful in cases of asset-only selection, since it leads to the same investment choices. A FSD downward shift implies the existence of the entire frontier so long as returns are normally distributed. SSD is consistent with that part of the frontier above the minimum variance point that is generated by means-variance. Even with the quadratic's limitations regarding normality and IARA assumptions, means-variance will still generate the same optimal set of allocations as does first and second order dominance procedures.

This argument only supports asset-side analysis, however. Once liabilities enter the picture, even this proposition loses much of its effectiveness. Two assets having the same SSD characteristics may be indeterminate, or may not be normally distributed (as is required for quadratic utility to hold). MVO analysis will be insufficient for a determination of which asset is third-order dominant. Means-variance only uses the first two moments of an asset's return distribution, without any analysis given to downside risk. Skewness in the probability distribution can create varying degrees of downside risk, while liability exposure that is external to the asset probability mass will also routinely exist. MPT cannot ascertain which of two assets on the frontier will be preferred, given the same FSD and SSD conditions but different third-order dominant positions or external liabilities.

With these concerns regarding the quadratic, power equations have come into widespread usage among utility analysts. The power series generates decreasing absolute risk aversion, and is very adaptive to changing relative risk aversion and return distribution assumptions.

An Expanded Process. When the classical utility assumptions are changed to more realistic observations, investor utility can quite conceivably increase, stay the same, or decrease in response to changing allocations across time. To identify the investment choices more clearly, let's first consider the basic investment choice between the risk-free asset and a risky asset. Using power utility, keeping the IID assumption for the moment, and then varying only the investor's relative risk aversion yields the following conclusions. Power utility is modeled by the following equations that were noted in the above discussion on utility functions: [19]

$$u(z) = (1 / (b-1))(a + b z)^{(1 - 1/b)}$$

Where, z is wealth and u (z) is the utility of the wealth z; b>0; and z > [-a / b, 0]. Absolute risk aversion is determined by:

$$A(z) = -u''(z) / u'(z) = 1 / (a + b z)$$

Relative risk aversion is then calculated as:

$$R(z) = -u''(z)\, z / u'(z) = z / (a + b z)$$

Changes in relative risk aversion are accomplished through the value of the "a" parameter. If a > 0, increasing relative risk (IRRA) exists. If a = 0, then CRRA occurs. A negative parameter setting will generate DRRA preferences. "b" is often set at a value of 2, although the value only changes the magnitude of the wealth, z.

With an IID assumption, constant relative risk aversion (CRRA) will produce the same investor utility over time. In this one instance, time is invariant to the process. Moving to IID and increasing risk aversion (IRRA), utility of wealth will decrease over time with the risky asset compared to the utility enjoyed with the risk-free asset. Investors will prefer the risk-free asset to a risky asset. Conversely, when IID and decreasing relative risk (DRRA) are assumed, the agent will prefer the risky asset to the risk free asset.

Varying the IID assumption in addition to varying the relative risk assumption leads to interesting and more complex results. Using the above equations, we can ascertain the impact of varying both the IID and relative risk assumptions.

Predictability of returns can either enhance or reverse the effect experienced with relative risk aversion. With constant risk aversion (CRRA) and return predictability, time will no longer be invariant. Instead, the investor will prefer the risky asset to the risk-free asset. Predictability of returns will enhance the preference for risky assets that an investor with DRRA preferences already possesses. Of great interest, predictability can be so large that it even reverses the normal effect that increasing risk aversion has upon asset allocation. Where predictability is large, the investor will prefer the risky asset to the risk-free asset even with increasing relative risk (IRRA).

Mean acceleration of returns will have the opposite effect of predictability. This is because return acceleration will vastly increase standard deviation across all holding periods (relative to the standard error estimate) while return predictability greatly reduces standard deviation across time (again, relative to standard error). Return acceleration can be so large in its impact that it can produce decreasing equity allocations even with decreasing risk aversion. It will also enhance the preference for the risk free-asset that an IRRA investor already possesses.

Predictability and mean acceleration can also occur with varying levels of intensity. As the intensity of predictability or acceleration increases, the impact upon allocation decisions will also increase. In fact, threshold levels can estimate the points at which mean reversion or acceleration will reverse a contrary effect from relative risk aversion.[20] Utility parameters can be set so high that the reversion and acceleration of returns reverses the effects that relative risk has upon allocations. This is consistent with the previously noted empirical tests upon return data, with return predictability being so large that huge equity allocations were generated in longer holding periods. For overall modeling purposes however, parameters could be set so that predictability or acceleration merely slows down, but does not completely reverse, a contrary effect stemming from opposite relative risk preferences.

In fact, any level of intensity of mean predictability and risk aversion can be modeled. Two different investors could both be decreasing in risk aversion for example, but one has greater aversion to risk relative compared to the other. The strength of the coefficient of risk aversion, γ, can make an important difference in the overall conclusions on the relevancy of time.

Gollier and others state the portfolio problem with the following analysis.[21] With predictability, LT horizon investors will take more risks early in life. A dynamic process is used, and equations involve return through each period, $x\sim_t$. Predictability exists when $x\sim_1$ is correlated to $x\sim_0$. $E\ x\sim_0 > 0$, due to predictability of returns. The value function is:

$$u\ (z, x_0) = \max_\alpha E\ [\ (z + \alpha\ x\sim_1\ x\sim_1)^{(1-\gamma)} /((1-\gamma)\ \big|\ x_0].$$

The function will be separable for each period, and the first period problem can now be solved via backward induction. From Gollier (Opt. Port. Mgt, 2005):

$$v_{n-1}\ (z, s_{n-1}) = \max_{c1\ldots cs} \Sigma^S_{s=1}\ p_s\ u\ (c_s)\ \text{ s.t. } \Sigma^S_{s=1} \prod\nolimits_s (s_{n-1})c_s = z$$

where the vector of prices in the last period depends upon states of nature s-1 that prevailed one period earlier. With CRRA > unity and mean reversion of returns, younger households should now have riskier portfolios. This is because the wealth effect dominates the precautionary effect, with overall marginal value of wealth increasing for riskier portfolios in the presence of predictability.

When the return distribution is not perfectly known, the optimal strategy is affected. This type of parameter uncertainty will tend to make the agent more conservative than with a known predictable return structure. This is consistent with Barberis (1999) and Reiss (2006). See the below discussion on allocations. In the early stages of learning of return distributions, it would therefore be prudent of the investor to be more conservative than at a later learning stage where more understanding exists as to return predictability. This was also noted in Gollier (Opt. Port. Mgt, 2005).

Allocations Using Power Utility. Modeling allocations through power utility has many advantages over traditional means-variance procedures. Power equations may be a more theoretically appropriate construct to use for time-horizon modeling than MVO, since absolute and relative risk as well as other utility and liability factors can be expressly accounted for.

Many studies have shown strong mean reversion of equity assets in longer time frames. Knowledge of return predictability can be quite useful in the allocation of assets, if a model can incorporate varying return predictabilities. Fortunately for us, power utility functions provide for varying amounts of return predictability as well as relative risk aversion. As an example of the possibilities, Barberis (2000) used power utility concepts and dynamic allocation equations to calculate optimal allocations. The investor's utility preference is described by a CRRA power utility function:

$$v\ (W) = W^{1-A} / (1 - A)$$

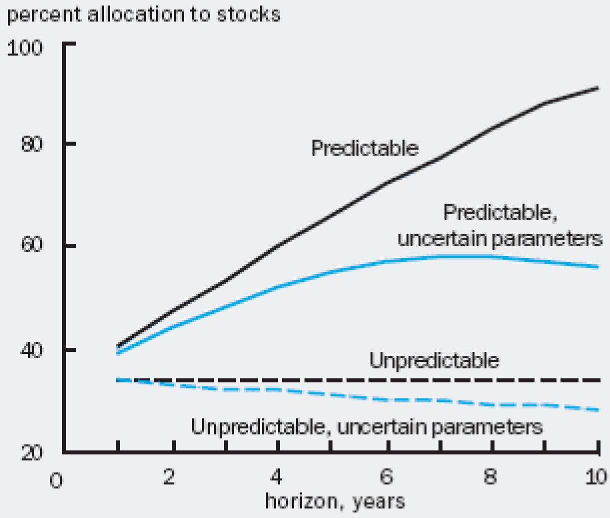
where W is wealth, and A is the risk aversion coefficient. Dynamic rebalancing occurs with:

$$\max_{to} E_{to} (W^{1-A}{}_{K} / (1-A))$$

The introduction of return predictability greatly increased equity allocations. When predictability of returns were made somewhat uncertain however, the same equations reduced the equity allotments, but still ended up with stock allocations that were higher than with no predictability being present at all. Parameter uncertainty is compared against certain returns, with uncertain returns being:

$$P\ (R_{T+T^\wedge} \mid r), \text{ where } r = (r_1 \ldots r_T)$$

The following graph displays the results for the assumptions of CRRA and a risk aversion coefficient of 10:[22]



Allocation to stocks, different investment horizons

Notes: The investor maximizes the utility of terminal wealth via a buy-and-hold investment in stocks versus bonds. The investor has constant relative risk aversion utility with a risk aversion coefficient of 10. The top calculation (black) includes predictable returns modeled by a regression on d/p ratios (equation 1). The second calculation (color) includes predictable returns and the effects of parameter uncertainty. The third calculation (black dash) assumes unpredictable returns, and no parameter uncertainty. The bottom calculation (color dash) assumes unpredictable returns, but adds parameter uncertainty. All distributions are conditional on a dividend/price ratio equal to its historical mean.

Source: Barberis (1999).

Following the above graph, for buy and hold investors, if returns are IID, then optimal allocations would be almost constant across time, at something under 40%. This result is strikingly similar to the time irrelevance conclusion of Samuelson, although Samuelson used periodic rebalancing versus the buy and hold investor of Barberis. If return parameters are uncertain with an IID assumption, then equity allocations would actually decrease slightly over time. Once returns become predictable, allocations increase to 100% equities by the ten-year hold. But when uncertainty is introduced with predictability, optimal equity allocations peak at around 60%, and actually decline somewhat in the very long holding periods. Barberis used CRRA with varying coefficients of risk aversion in his calculations. A risk aversion coefficient of 10 was used to generate the above graph. If risk aversion would be higher or lower at the individual level, then corresponding changes would occur to optimal allocation levels. For investors who optimally rebalance the portfolio in a dynamic setting, investors would also allocate substantially more to equities, the longer the horizon. Higher equity allocations give investors a hedge in investment opportunities.[23]

Other researchers have also used power utility with constant relative risk aversion to produce similar results. For instance, one recent study evaluated the effects of predictability and return uncertainty.[24] The model with predictability and expectations risk added is:

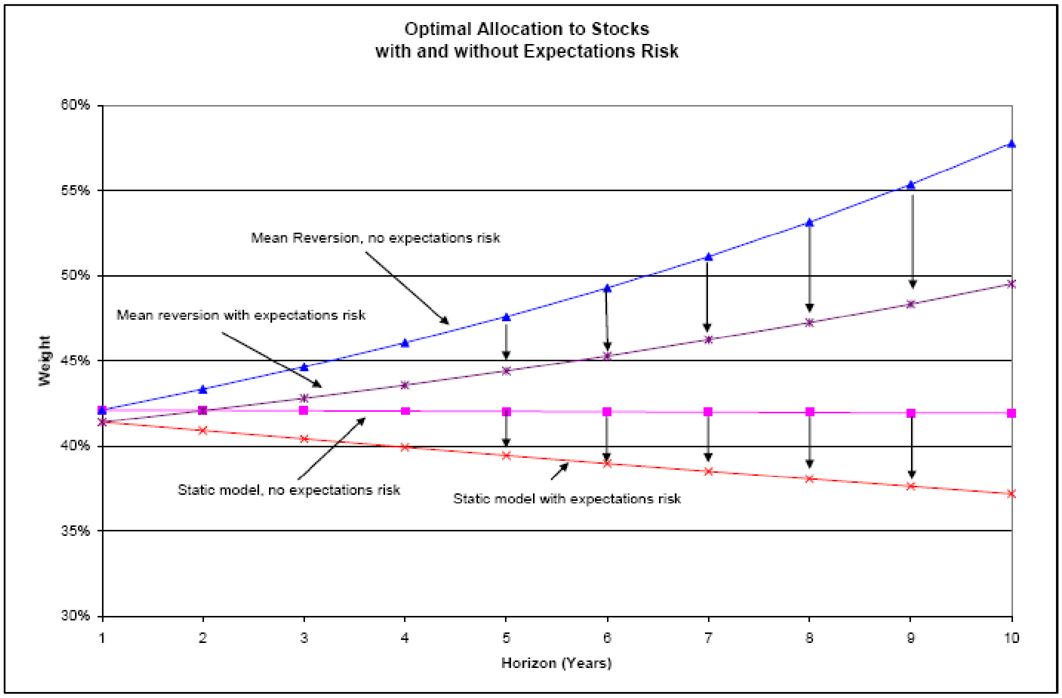
$$R_t = \mu e + \varepsilon m_t + (be + \varepsilon b) X (d_t - D) + \varepsilon r_t,$$

where μe is our estimate of the mean return ($\mu = \mu e + \varepsilon m_t$), and εm is a random variable with a mean of zero and standard deviation of σm, representing the error in our estimate. d_t is the dividend yield at time t, D is the "normal" dividend yield and b is the impact of dividend yield on return; be is the estimated value of b, and εb is the error in that estimate with standard deviation σb. The T-year variance is:

$$T\ \sigma r^{2} + T^{2}\ \sigma m^{2}$$

where ρm represents the correlation in errors in the mean from period to period, with a standard deviation of σr. These optimizations maximize expected utility using the Constant Relative Risk Aversion (CRRA) power utility function $U(W) = W\ (1- \gamma\ ) - 1)/ (1 - \gamma\ )$ with $\gamma = 4$.

With independent returns, optimal allocations were estimated at 43% equities with no return uncertainty, but declined over a 10-year period to 37% equities with expectation risk. Then, when return predictability was introduced, optimal allocations increased to 75% equities at 10-years, but only 50% allocations with predictability and estimation risk. The following graph is indicative of the optimal allocations. Notice the similarity to the Barberis graph, above.



These examples show that optimal allocation of assets depends upon a number of utility related factors. In addition to the mean and variance of asset returns, the type of utility function chosen, the time frame of investment, investor-level risk aversion, the presence of return predictability, parameter uncertainty, and the usage of dynamic programming all have a direct bearing on the allocations ultimately selected.

Instead of viewing the investment process as a balancing between merely pricing return and pricing volatility, the allocation decision can be viewed as a balancing of various utility factors, including return predictability and investor-level risk aversion. This is especially the case when power utility is used. The following table shows the impact on allocations from the continuum of investor-level choices within a general utility context.[25] The table can be further refined to incorporate the impact from the other investor-level factors, including labor income shocks variability, parameter uncertainty, and changes in the risk free rate of return.

Optimal Asset Allocations Across Time

	IID	**Predictable**	**MA**
IRRA	Decreasing	Increasing	Decreasing
CRRA	Constant	Increasing	Decreasing
DRRA	Increasing	Increasing	Decreasing

With extended power utility, identical and independent returns (IID) and constant risk aversion, investors would have fixed allocations across time. The results of these classical assumptions represent only one-ninth of the continuum of assumption choices in the above graph, but were key assumptions in early studies which concluded that time does not matter in investment policy. When quadratic types of assumptions are made, such as IRRA and IID, allocations decrease, but when the quadratic combines with return predictability, the result could be increasing allocations. A long-term investor typically would have DRRA and predictability preferences, resulting in increasing allocations across time.

If the investor has a time horizon and trading style (i.e. long-term buy and hold) that enables him to take advantage of return predictability, equity allocations would increase with constant relative risk aversion preferences. DRRA investors would accelerate their normal preferences for increasing equity allocations over time. Investors with IRRA preferences could have varying responses to return predictability. As mentioned previously, empirical tests suggest that the effect of predictability is so large that it could completely reverse the decreasing equity allocation tendencies of IRRA investors. If the investor does not take full advantage of predictability however (i.e. with a short or near-term style), then the intensity of the predictability will be lessened. Also, the investor may have varying intensity concerning risk aversion. Conceivably, the IRRA investor could experience constant allocations or even retain the decreasing equity preferences, but only at a slowed rate.

Mean acceleration of returns would work in reverse. CRRA investors would move from constant to decreasing equity allocations, while IRRA investors would increase their normal preferences to decrease equities across time. This is a natural response to the exploding pricing volatility across all time frames that is presented by mean acceleration. The effect could be so tremendous that even investors with DRRA preferences could reverse their typical preferences for increasing equity allocations over time. If the effect of mean acceleration is less intensely experienced by the investor, then the natural tendency of the DRRA for increasing equity allocations could continue, but only a slowed pace. Or, the momentum of returns could completely offset the DRRA preferences, resulting in constant allocations across time.

Concluding Remarks. When utility assumptions are constrained to constant relative risk and independent return distributions, conclusions of time irrelevance from the classical multi-period models should actually be expected. It is only when these precise assumptions are used, in fact, that time is unimportant to the investing experience. When these theoretically self-imposed constraints are relaxed, a continuum of economic choices under conditions of uncertainty emerges.

Moving away from quadratic utility and towards power utility allows absolute risk aversion to be better emulated. Varying the relative risk assumption then brings shifting investor-level risk preferences into the portfolio modeling process. Return predictability allows investors to take advantage of reversion to the mean tendencies occurring in most asset classes. By looking at the full opportunity set of utility factors, including income shocks, liquidity constraints, and parameter uncertainty, investment managers can determine the limits that risk aversion and predictability have upon portfolio and net worth composition.

Utility theorists use dynamic programming in the calculations of all of these variables and concepts. The output provides strategies and benefits far in excess of what is available in a timeless setting. The diversification of assets in any one time period and the allocation of assets across all time periods provides tangible advantages for a philosophy of long-term investing. Investors adjust short-term allocations in an effort to smooth consumption over time and maximize lifetime consumption. A coherent, economic theory of risk and return across all time frames finally explains, and is consistent with, the long-standing practices of investment management.

This deliberate adjustment of resources over a person's lifetime is simply not contemplated in the special case of utility. Dynamic portfolio management thus provides a more complete understanding of the possible long-term economic benefits available to investors and consumers. As Gollier (2001) noted, dynamic modeling can certainly mimic MVO output by assuming IRRA. Samuelson's time irrelevance belief can also be simulated with CRRA and IID assumptions. The dynamic management of net wealth merely increases the economic choices of agents. The investor effectively has different relative risk options to choose from; the return probability mass can have varying states of mean reversion, independence, or mean aversion; the intensity of risk aversion and predictability can be estimated; and other utility factors that might limit or reinforce allocation decisions can also be considered. By including all of these variables into a general utility setting, the investor's choices in the face of continuing uncertainty and risk expands considerably from that of the timeless, means-variance environment.

Dynamic efforts have many theoretical advantages over what is described as "modern" portfolio theory. In the coming years, investment professionals would secure a large economic advantage for their clients and beneficiaries by moving to general utility and dynamic processes. To be sure, the methods are more complicated. And so far, Expected Utility has not produced the kinds of uniform and standardized procedures that are available with means-variance optimization. In so many ways however, investing is utility. The finance community would do well by adopting all of the available analytical tools for use in the calculation of utility across all intended holding periods.

Appendix - Literature Review

Phelps (1962) was one of the earliest articles on the subject.[26] The paper used a lifetime consumption strategy with a stochastic, discrete time programming model where an investor is exposed to the risk of a loss. The utility function is:

$$U = \sum^{N}_{t=1} \alpha^{n-1} u(c_n)$$

where c_n is consumption of capital. The consumption sequence is IID. u (c_n) is bounded from above and below, and is a strictly increasing and concave function. Consumption is an increasing function of age and capital. Using a utility function of u (c_n) = u upper bound – $\lambda\ c_n^{-\gamma}$, a linear consumption function is generated in capital and non-wealth income. (KCK note: this is a HARA equation that produces a linear consumption; double the income or capital, double the consumption; this will generate a constant portfolio rule). As Phelps states at 741, "risk and return valuations have opposing qualitative effects upon consumption".

Another utility function is considered: u $(c_n) = \lambda\ c^{\gamma}$. Both utility functions are "constant-elasticity utility functions with elastic parameter γ." (Phelps, at 742). Log utility is also reviewed, u (c_n) = log c_n. The optimal consumption rate will be independent of expected return and riskiness of capital. Consumption is linear in capital.

This article was an early "exploratory effort" at using von Neumann-Morgenstern expected utility. It established the tone for further research by Samuelson and Merton in that all utility functions reviewed were either HARA or log in nature, resulting in an optimal consumption rate that is independent of expected return or risk. The paper did not state a time invariant decision rule, but did note the linear nature of consumption to wealth and capital.

Samuelson (1963) wrote a classic article.[27] It was shown that if a person refuses to take favorable odds on a single toss, then the person must rationally refuse to participate in a sequence of such tosses. The aggregate of single events will generate the same expected utility as that of the single event. In the St. Petersburg Paradox, risk could be reduced by increasing the number of ship owners that are facing the risk of loss.

Samuelson's 1963 paper is considered a classic in utility circles. It established a general recognition that risk still exists in large numbers. Improbable events could still occur. On an interesting note applicable to time modeling, Samuelson assumed a fair bet that was IID in his proof.

Samuelson (1965) was another very important article.[28] It was shown that there is no way of making an expected profit by extrapolating past changes in the futures price, by "chart or other esoteric devices of magic or mathematics". The market quotation of Y(T,

t) already contains in itself all that can be known about the future. It has all the discounted future contingencies in it. The fair-game pricing is:

$$E\,[\Delta^{n}\,Y\,(T,t)] \equiv 0 \quad (n = 1, 2, \ldots T)$$

With bias, the drift in pricing would be $E\,[\Delta\,X] = 0$, and thus, $E\,[\Delta\,Y] = 0$. If there is a biased drift in pricing, then $E\,[\Delta\,Y] = [\Delta\,X] + u\,(T - 1 - T) = u - u = 0$, and we are back to $E\,[\Delta\,Y] = 0$.

This 1965 paper was seen at the time as a refutation of technical analysis. It was highly influential in developing basic tenets of efficiency, demonstrating that randomness in security pricing did not contradict the laws of supply and demand, and that the market price contains all that is or can be known about its future contingencies. Samuelson cited the paper when he wrote on time irrelevance in 1969.

Samuelson (1969) became a core paper on the usage of time horizons in investment analysis.[29] Uses log utility, log C is "interesting and admissible" (at 243). A CRRA assumption was also made. A multi-period model was generated in discrete time. Samuelson supplied proof that for an isoelastic marginal utility function, $U'(C) = C^{\gamma-1}$, $\gamma < 1$, the optimal portfolio decision was independent of all consumption savings decisions, leading to a constant w*, or percentage allocation. Samuelson used a no-bequest assumption, but noted that Merton's companion article (1969) comes up with similar conclusions with bequests factored into the last period.

The two 1969 papers of Samuelson and Merton may have been the first articles to expressly state a time irrelevant result. They were enormously influential in establishing the belief that long horizons do not result in changing allocations at optimality. The Samuelson paper should be seen as only proving that log utility produces a time invariant result, however. Note that the Elton, Gruber, et al shows that the log function of the general type of: $U(w) = \log(w)$ will exhibit DARA and CRRA. Kritzman and Rich indicate that log utility functions will produce a time invariant result for a CRRA assumption, regardless of predictability.

Merton (1969) is the companion paper to Samuelson's 1969 effort.[30] Merton developed a continuous time model in the article. Merton useda Bellman principle to derive optimality. Merton also assumed CRRA ("iso-elastic marginal utility"). Where $U(C) = \log C$, and $U\,(C) = = C^{\gamma} / \gamma$, $\gamma < 1$, and $\gamma = 0$. The portfolio problem was generally stated as: $\mathrm{Max}_{\{C,\,w\}}\ \Phi\,(w, C; W; t) = 0$; where, w is the proportion invested in the risky asset; C is consumption. W is total wealth; and t is time; Merton states the optimal consumption and portfolio selection rules as (at 251).

$$c^{*}\,(t) = [\,v / (\,1 + v\,\varepsilon - 1)\,e^{v\,(t-T)}]\,W(t), \text{ for } v <> 0$$
$$c^{*}\,(t) = [1/\,(T - t + \varepsilon\,)]W\,(t), \text{ for } v = 0;$$
$$\text{and } w^{*}(t) = (\alpha - \gamma) / \sigma^{2}\,(1 - \gamma) \equiv w^{*}$$

where, w (t) is the proportion invested in the risky asset at time t; c (t) is consumption per unit at time t; c* and w* are consumption and the risky proportion at equilibrium or optimality; W (t) is the total wealth at time t;

Merton stated that the optimal portfolio rules for an infinite horizon model is that for iso-elastic marginal utility, the portfolio decision is independent of the consumption decision. "The price change and the resulting level of wealth, have zero relevance for the portfolio decision and hence it is constant" (at 253). The optimal proportion of the risky asset, w*, can be rewritten in terms of Pratt's RRA measure, δ as: $w^* = (\alpha - \gamma) / \sigma^2 \delta$. The optimum composite portfolio will have a constant mean and variance.

Individuals with low RRA, $0 < \delta < 1$, will choose to consume less now and save more to take advantage of the higher yield available, and the substitution effect dominates the income effect. Where $\delta = 1$, the income effect just cancels out the substitution effect. For risk averters, $\delta > 1$, the income effect dominates the substitution effect.

Merton also looked at CARA utility. This function is exponential in nature, but with CARA and CRRA characteristics:

$$U(C) = - e^{-\eta C} / \eta, \eta > 0$$
$$\text{then: } - U''(C) / U'(Cc) = \eta,$$

which is Pratt's ARA measure. Consumption is no longer a constant proportion of wealth (as is the case with log utility), but it is still linear with wealth. Instead of the proportion of wealth invested in the risky asset being constant ($w^*(t)$ = constant), the total dollar value of wealth invested in the risky asset is constant ($w^*(t)\ W(t)$ = constant).

In the paper, Merton extended the Samuelson time irrelevance conclusion into continuous time. The two papers laid the theoretical groundwork for a unitary time horizon assumption in MPT. This paper showed that log utility with a CRRA assumption in continuous time generated a constant portfolio rule. The paper also showed that CARA utility was time invariant as to the total dollar value of wealth invested in the risky asset being constant. While the impact of the paper was significant at the time, the paper does not cover power utility, and is essentially limited to log utility with CRRA assumptions or CARA utility.

Fama and Miller (1971) contained an entire chapter on multi-period consumption investment models.[31] It was based in large part on a 1970 article by Fama.[32] Dynamic programming, with backward induction is a natural approach (at 324). A discrete time multi-period model is:

$$U_t (C_{t-1}, w_t \mid \Phi_t) = \max_{ct, h} \int_{\Phi t+1} U_{t+1} (C_t, \textstyle\sum^N_{j=1} h_f [1 + Rj (\Phi_{t+1}]) \Phi_{t+1}\, dF_{\Phi t} (\Phi_{t+1})$$
$$\text{s.t } 0 <= C_t <= w_t \quad \text{and } \textstyle\sum^N_{j=1} h_f = w_t - c_t;$$

and where c = consumption in time t; and w is wealth in time t; and Φ is the "state of the world". $U_t (C_{t-1}, w_t \mid \Phi_t)$ is the maximum expected lifetime consumption. The process is a sequence of two-period problems that are solved to maximize lifetime consumption. The authors provide a theorem that a strictly concave function (i.e. risk aversion) of lifetime consumption will have all derived functions in time t having the same properties. The authors also use a two-period two-parameter model with mean and variance that was previously developed in earlier chapters of the book.

The text was used as instructional material at the University of Chicago. The model is an excellent example of early efforts using dynamic programming in inter-temporal models. Risk aversion was generally assumed through a concave utility function. References were also made to a two parameter model of mean and variance. With Fama's and Miller's usage of dynamic programming, the modeling of assets across time definitely became more sophisticated than a single-period model.

Merton (1973) continued his thoughts from the 1969 paper, writing an entire book on what he referred to as the "Intertemporal Capital Asset Pricing Model", or ICAPM. [33] The Campbell (1993) preface to the text stated that for the last 25 years, an important goal of financial research had been to generalize the insights of the single-period CAPM to a multi-period setting. This has been difficult to achieve because the multi-period consumption and portfolio choice problem is inherently non-linear. Merton (1969, 1971, 1973) placed the problem in continuous time. Doing this effectively linearizes the results by making the decision interval infinitely small, with the model being linear over the interval. This kind of linearity is only local in nature.

(Comments by a Virginia Tech paper):[34] The ICAPM replaces dependence on quadratic utility/normal returns with the assumption of a GBM process which implies normally distributed returns. In the continuous time setting, higher moments do not matter, improving tractability of the model. An advantage over the CAPM is utility can be state-dependent, although the time-separability assumption remains.

Utility is estimated over lifetime consumption and rewritten as current utility plus this indirect utility function, J(W). That is, J(W) represents the optimized value of all the future utility. Where $J_{Wx} = 0$, investors do not want to hedge (log utility). Note that

$$J(W, t, X) = \max E_t \left[\int_t^T U[C(s), s] ds + B[W(T), T] \right] .$$

instead of the traditional CAPM assumptions of quadratic utility or joint normality of returns, we instead use an Ito process for returns. This has the implication that only the first two moments matter, putting us in the mean-variance world. The indirect utility function is:

Optimal Portfolio weights are:

$$\mathbf{w}^* = \left(\frac{-J_W}{W J_{WW}} \right) \mathbf{\Sigma}^{-1} \boldsymbol{\alpha}^e + \left(\frac{-\sum_{i=1}^{M} J_{W x_i} g_i}{W J_{WW}} \right) \mathbf{\Sigma}^{-1} \boldsymbol{\sigma}^{\eta_i}$$

The model can also be used to minimize the variation in consumption growth. This provides a prelude to consumption based asset pricing models of Breeden (1979) and others. Optimal weights become:

$$\mathbf{w} = \lambda / W^2 \mathbf{\Sigma}^{-1} \boldsymbol{\alpha}^e = k\mathbf{t}$$

With constant risk tolerance utility functions and constant investment opportunities, optimal portfolio choices are also constant. When the investment opportunity set changes, so will portfolio allocations. A single state ICAPM can be written as:

$$\alpha_i = r_f + \beta^m_i (\alpha_m - r_f) + B^h_i (\alpha_h - r_f)$$

where $\beta^m_i (\alpha_m - r_f)$ is the market risk premium and $B^h_i (\alpha_h - r_f)$ is the hedging component for uncertain future events affecting the opportunity set. Each additional state of a multi-beta model would then have a term like $B^h_i (\alpha_h - r_f)$ for each state variable. The standard CAPM effectively becomes a special case of the ICAPM, limited to situations where investors do not or cannot hedge against future adverse events and where no additional state variables come into play. The hedging component would drop out, leaving the CAPM's single-factor equation. Also note that instead of quadratic utility with the CAPM and MVO, the ICAPM uses an Ito process for returns. This results in only the first two moments mattering, with the model being in means-variance space.

(KCK Comments on Merton's 1973 text): Believing that current demands are affected by the possibility of uncertain changes in future investment opportunities, Merton showed that changes in interest rates produces changes in the investor's opportunity set. Merton's theorem was that all risk-averse investors will be indifferent between portfolios composed of three funds, namely the two funds traditionally associated with the mean-variance efficient frontier, plus a third one to hedge against unfavorable intertemporal shifts in the frontier. At equilibrium, investors will be compensated for bearing systematic risk and for bearing risks of unfavorable shifts in the opportunity set. Thus, the optimal portfolio chosen at the initial date will not only be on the efficient frontier, it must also provide the best hedge against future changes in the opportunity set.

Breeden (1979) developed a consumption based model.[35] (Comments from Va Tech paper, Fall, 2007): The CCAPM is very much like the ICAPM with consumption growth as the single state variable. In the ICAPM investors hedge against changes in the state variables because these represent changes in the investment opportunity set, and therefore, changes in consumption. The CCAPM goes directly to hedging against changes in consumption. The standard CAPM also has this consumption interpretation since it is a single period model so end of period wealth is the same as consumption. A key assumption in the CCAPM is additively separable preferences, which gives state independence of direct utility.

To make more clear the link between the ICAPM and the CCAPM, note that in the ICAPM agents set the marginal utility of wealth equal to the marginal utility of consumption along the optimal consumption path. This is the envelope condition, UC = JW. If markets are complete, then perfect hedges for the state variables can be formed and all individuals will have perfectly (instantaneously) correlated consumption policies. This is an analogue to all individuals holding the market portfolio in the static CAPM.

In many ways, the CCAPM is the most fundamental of the equilibrium models. It is illogical to choose the CAPM or ICAPM because you think the consumption-based model is wrong. The only reason for choosing an alternative model is because the consumption data to test the model may be unsatisfactory.

The combination of portfolios h and t which the investor chooses minimize the variance in consumption, not wealth. Defining a reference portfolio C:

$$\sigma_C^2 = \mathbf{w}_C' \boldsymbol{\sigma}_{rC} = T(\alpha_C - r_f).$$

Solving for T and substituting:

$$(\alpha_i - r_f) = \frac{\sigma_{iC}}{\sigma_C^2}(\alpha_C - r_f) = \beta_{iC}(\alpha_C - r_f).$$

Note that if the consumption portfolio is not itself a traded asset than the portfolio with the maximum correlation with consumption can be used. The same basic intuition applies, but this results in the same kind of instrumental variable as in the previous presentation of the ICAPM. If consumption is available, it serves as the single variable driving the returns process. When it is not available we include additional state variables to use as instruments.

John Campbell (1993) solved the portfolio problem in a closed form. [36] The inter-temporal budget constraint is loglinear, allowing the consumption and portfolio problem to be solved in a closed form. This avoids the necessity of assuming that the time interval is small (Merton, 1969, 1971, 1973). Thus, the consumption-wealth ratio is small. Consumption based models (i.e. Breeden, Lucas, Shiller) have generally rejected such restrictions. The model presented here allows for empirical testing without consumption data. The dynamic budget constraint is:

$$W_{t+1} = R_{m,\,t+1}(W_t - C_t)$$

Where W = total wealth, including human capital; C is consumption; R is the gross simple return on wealth invested from t to t+1. The “m” denotes wealth invested in the market portfolio. The loglinear approximation is:

$$\Delta w_{t+1} = r_{m,\,t+1} + \log(1 - \exp(c_t - w_t))$$

Non EU model proposed by Epstein and Zin (1989, 1991) is used to generate a loglinear Euler equation.

Nicholas Barberis (1999) focused on allocation decisions using power utility.[37] On the theoretical side, it has been known since Merton (1973) that variation in expected returns over time can potentially introduce horizon effects. The model uses power utility in discrete time with two assets, T bills and a stock index. A VAR model is used, and parameter uncertainty is introduced by a Bayesian approach. A comparison is made between optimal rebalancing, buy and hold investors, and dynamic rebalancing.

For initial wealth of $W_T = 1$; w = equity allocation; T^ = number of periods, the end horizon wealth is:

$$W_{T+T\hat{}} = (1 - w) \exp (r_f T\hat{}) + w \exp (r_f T\hat{} + r_T + \dots r_{T+T\hat{}})$$

The investor's utility preference is described by a CRRA power utility function:

$$v (W) = W^{1-A} / (1 - A)$$

where A is the risk aversion coefficient. The investor then solves for:

$$\max_w \int v (W_{T+T\hat{}}) \, p (R_{T+T\hat{}} \mid z) \, dR_{T+T\hat{}}$$

Parameter uncertainty is compared against certain returns, with uncertain returns being:

$$P (R_{T+T\hat{}} \mid r), \text{ where } r = (r_1 \dots r_T)$$

And certainty is expressed as: $P (R_{T+T\hat{}} \mid r, u, \sigma^2)$. Predictability is then analyzed with a vector z_t that contains two components: the excess stock return r_t, and a predictor variable, $x_{1,\ t}$. This variable is composed of the dividend yield is used to capture the variation in expected returns. The investor then solves for lifetime consumption.

Dynamic rebalancing is introduced with:

$$\max_{to} E_{to} (W^{1-A}{}_K / (1-A))$$

Where, $\max_{to}$ means that the investor maximizes over all remaining decisions from time t_o forward. W is derived as:

$$W_{k+1} = W_k\{(1 - w_k) \exp (r_f T\hat{} / K) + w_k \exp (r_f T\hat{} / K + R_{k+1})\}$$

The utility function now becomes:

$$J (W_k, x_k, t_k) = \max_{tk} E_{tk} \{W^{1-A} / (1 - A)\}$$

(Conclusions, at 24 of the working paper): If there is certainty as to parameters, a buy and hold investor would invest substantially more in risky assets in the presence of predictability, the longer the horizon. Time variation in expected returns induces mean reversion, slowing the growth of variances of long horizon returns. This makes equities appear to be less risky to long-term investors. In a dynamic setting with optimal portfolio rebalancing, an investor more risk averse than a log utility investor would also allocate substantially more to equities, the longer his horizon.

Investment advisors often maintain that long-run investors should allocate more aggressively to equities. This view finds little support in a world with i.i.d. asset returns, a point made forcefully by Samuelson (1969). The results presented here suggest that time-variation in asset returns may provide a rationale for practitioners' recommendations after all.

Portfolio calculations can be seriously misleading if the allocation framework ignores the uncertainty surrounding parameters however. When this source of uncertainty is accounted for, long-horizon investors in general still allocate more to equities than short-horizon investors, but the difference is not as large. In some cases, the estimation risk can be so severe as to make the optimal stock allocation *decrease* with the investor's horizon. Moreover, parameter uncertainty makes the optimal allocation much less sensitive to the current value of the predictor. This suggests that analyses which ignore estimation risk may lead the investor to take positions in stocks which are both too large and too sensitive to the predictor. Barberis (at 244) contains several graphs showing allocation effects. Cochrane (1999) also has graphs from Barberis, reformatted for color.

Stangeland and Turtle (1999) was one of many articles reviewing utility concepts following the success of Siegel's *Stock for the Long Run* (1994, 1998).[38] Siegel advocated strong equity allocations over long time frames, showing standard deviation is greatly reduced in long time periods. See, Kaufhold, the Time chapter of the draft book, *A Philosophy of the Long-Term*, for an extensive discussion of Siegel's method. Generally, articles responding to Siegel attempted to sort out various utility-related concepts. Power utility is modeled by:

$$u(z) = (1 / (b-1))\,(a + b z)^{(1 - 1/b)}$$

Where, z is wealth and u (z) is the utility of the wealth z; b>0; and z > [-a / b, 0]. Absolute risk aversion is determined by:

$$A(z) = -u''(z) / u'(z) = 1 / (a + b z)$$

Relative risk aversion is then calculated as:

$$R(z) = -u''(z)\, z / u'(z) = z / (a + b z)$$

Changes in relative risk aversion are accomplished through the value of the "a" parameter. If a is positive, increasing relative risk (IRRA) exists. If a = 0, then CRRA occurs. A negative parameter setting will generate DRRA preferences. “b” is often set at a value of 2, although the value only changes the magnitude of the wealth, z.

The equation is very useful, as varying RRA preferences and varying conditions of predictability can both be modeled simultaneously. RRA depends upon the sign of a. DARA with IRRA will be produced if $a > 0$; DARA with CRRA if $a = 0$; and DARA with DRRA if $a < 0$, all subject to $z >$ max of $(-a/b; 0)$. With this equation, IRRA and IID will have lower utility than the utility of the RFRR; CRRA and IID has no change in utility compared with RFRR; and DRRA and IID has greater utility than with the RFRR. Of great interest, the RTM (and MA) effects can be so large that RTM has greater utility than the RFRR with all forms of RRA (even IRRA). The effects of MA can be so large that there is less utility than the utility RFRR with all RRA (even DRRA). Thus, predictability can enhance the effect of RRA (as with DRRA and RTM) or reverse the effects of RRA (as with MA and RTM).

Predictability and mean acceleration can also occur with varying levels of intensity. As the intensity of predictability or acceleration increases, the impact upon allocation decisions will also increase. In fact, threshold levels can estimate the points at which mean reversion or acceleration will reverse a contrary effect from relative risk aversion. Utility parameters can be set so high that the reversion and acceleration of returns reverses the effects that relative risk has upon allocations. For overall modeling purposes however, parameters could be set so that predictability or acceleration merely slows down, but does not completely reverse, a contrary effect stemming from opposite relative risk preferences.

Campbell and Viceira (2002) is an involved life-cycle model.[39] The optimal strategy would involve periodic rebalancing in the face of changing forecasting variables, dividend-payout ratios, nominal interest rates, and projected excess risky returns. Risky asset portfolio percentages could be leveraged (or shorted), depending upon whether excess returns or positive (or negative). The book is considered a theoretical solution to the market-timing question, only with lifetime consumption being modeled instead of a short-term fixed horizon. The following analysis concentrates on the life-cycle model itself, and is taken from Kaufhold, draft book, *A Philosophy of the Long-Term.* .

With a power utility function, an agent’s multi-period consumption can be mathematically described as:

$$C_{kt}^{(1-\gamma)} / (1-\gamma) = E_t \, \Sigma_{I=1}^{T-1} \, \delta^i \, (\Pi_{j=0}^{i-1} \, p_{t+j}) \, (C_{k,\,t+1}{}^{(1-\gamma)} / (1-\gamma))$$

where, C is the level of consumption on date t; γ is the coefficient of relative risk aversion which is assumed to be > 0; the discount factor is δ, and is assumed to be < 1; p_t = probability that investor is alive at date t+1. Other assumptions and conditions also apply.

In a dynamic setting, the optimal portfolio problem is:

$$\max Eu = \int U(C, t)\, dt$$

Consumption toady is achieved at the cost of consumption tomorrow, and there is a balance between them. Optimality is a function of the risk premium, inversely proportional to the volatility and relative risk aversion, and then the hedging component. This is stated as a Bellman equation:

$$U(C) = J\ w$$
$$A\sim = 1 / \{(-J_{ww}\ W / J_w) * ((u_p - r) / \sigma^2_p) - (J_{ws} / J_{ww}\ W) * (\sigma_s / \sigma_p)\ \rho_{ps}\}$$

The investor will attempt to maximize the above equation, subject to several additional work-life and retirement budget constraints. The solution is obtained via backward induction.

Campbell and Viceira demonstrate that consumption is linked closely to net income, with little savings occurring outside of retirement accounts until after age 40. Consumption rises with income early in life because of borrowing constraints, and then falls later in life. Consumption paths are not completely smoothed, with an increase early in life, followed by a decrease in the later years. Total wealth and financial wealth is distinctly humped shaped, with a peaking occurring in the 1st year of retirement, and then sharply trailing off from there.

There are also extensive comments regarding human wealth (Id, ay 162-169). Human wealth can be considered a form of investment, being the PV of the future labor earnings. This form of wealth is non-tradable. RFRR has a constant log return rf per period, or rf ≡ log(1 + Rf). Stocks have log returns of rt+1 / period, and constant excess log return of $E_t\ (r_{t+1} - rf) \equiv \mu$, and variance $\sigma^2 \equiv Vt\ (rt+1)$. (at 162 et seq).

$$\text{Total wealth} = Wt + Ht,$$

where Wt is the financial wealth and Ht = labor wealth. Unconstrained,

$$\alpha\sim = (\mu + \sigma^2/2)/\ (\gamma\ \sigma^2)$$

where γ is RRA. The risky asset, stocks will be invested as: $\alpha\sim$ (wt + Ht), and the riskless asset (assuming two assets, one risky and one riskless) would be: (1- $\alpha\sim$) *(Wt + Ht). The implicit holdings in Ht become the riskless asset, and thus Wt should be adjusted so that total wealth has the unconstrained levels of the riskless asset. To this extent, Ht just substitutes for the RFRR. Stocks will have: $\alpha\sim$(wt + Ht), and the riskless asset will be (1- $\alpha\sim$) *(Wt + Ht) – Ht. Optimality will be:

$$\alpha = \alpha\sim(Wt + Ht) / Wt = (\mu + \sigma^2/2)/\ (\gamma\ \sigma^2)\ (1 + Ht / Wt).$$

α will now be > $\alpha\sim$ since Ht and wt are nonnegative.

Early in work life, the ratio Ht / Wt will be very large, for two reasons: 1) the agent will have large amounts of PV of future labor income; and 2) little or no financial wealth.

Over the work life, the Ht remaining will decrease, while wt will typically increase. This lowers the ratio over time until it is zero at retirement. Because this ratio is an optimal allocation equation, the share of total wealth devoted to risky assets will decrease over time.

This shows that an agent with riskless, nontradable labor income (ie stable, idiosyncratic income having no likelihood of variability) should weight the financial portfolio towards stocks more so than an agent who only owns tradable assets, and has no labor income streams. This is an important observation, since it tends to explain why younger people with stable employment will tend to be heavily concentrated in equities, while older people at retirement and with no labor income at all will be more conservative in risky allocations.

In reality, labor earnings are uncertain for many agents, making the human investment risky. As the variance of labor income increases, the tilt towards risky assets decrease. At the limit, the allocation of risky assets approaches that of a retired agent with no labor income. Starting with rf ≡ log(1 + Rf); and with the risky asset being a constant expected log excess return Et(rt+1 – rf) ≡ u. Then:

$$\max E_t (\delta (C_{t=1}^{1-\gamma} / (1-\gamma))) \text{ s.t. } C_{t+1} = W_t (1+R_{p,\, t+1}) + L_{t+1}, \text{ with } R_{p,\, t+1} = \alpha_t (R_{t+1} - Rf) + Rf.$$

The optimal allocation to the risky asset has two components, the first describing the uncorrelated labor income risk. The 2nd component is an income hedging item, where the risky asset desirability depends upon the ability to hedge against a bad realization of labor income. Optimality of the financial risky asset will now be:

$$\alpha_t = (1 / p) ((u + \sigma_u^2 / 2) / (\gamma \sigma_u^2)) + (1 - 1 / p) (\sigma_{Lu} / \sigma_u^2)$$

where the wealth elasticity of consumption is "p". The first component on the left side of eth + sign is the labor income risk, and the right side of the + side is the hedging component. If the covariance between the risky asset and the labor income is negative, the risky asset offers a good hedge against income shocks, and would increase the optimal allocation. Where labor income is idiosyncratic ($\sigma_{Lu} = 0$), the 2nd component drops out, and the introductory material then applies, with Ht substituting for the RFRR. Risky allocations of the financial assets would thereby increase.

The text analyzes the question of variability of labor income in terms of background risk. The conclusions of the analysis regarding background risk apply here, too. The increase in variance of labor income is assumed to be a mean-preserving increase. We also assume that labor income is uncorrelated to equities. Optimality becomes:

$$\alpha_t = (1 / (p (l, r_p))^* ((u + \sigma_u^2 / 2) / (\gamma \sigma_u^2))$$

Sufficiently conservative agents with risk aversion > 1/ p should reduce their financial exposure to risky assets in the presence of labor income variance. If labor income is correlated with risky returns, optimal risky allocations decrease. Where there is a completely positive correlation between labor income and risky assets (massive ownership of an employer's stock, for example), the agent should tilt away from the risky asset, since human investment becomes an implicit investment in that risky asset.

The crucial variables for purposes of asset allocation are liquid wealth, retirement wealth and future labor income. Labor income is observed to act as a substitute for the RFRR, with only a weak correlation between labor income and risky financial assets. Early in life, the investor will want to invest fully in equities, to the extent that he or she can. Borrowing constraints and a lack of liquid reserves will limit broader access to the financial markets at this stage. From age 40 onward, liquid wealth increases relative to future labor income, so that a shift in allocations gradually occurs away from risky financial assets.

Agents with greater precautionary tendencies will consume less and save more until retirement. Once at retirement, these investors will then consume more, since there is no longer any labor income risk to contend with, and retirement wealth may also be in a riskless annuity. Highly risk averse investors may have a lower portfolio allocation devoted to equities. Interestingly, the model predicts that such risk averse agents may still increase their equity percentage as they age, but will start from lower initial level of equities. For impatient households, consumption is greater early in life, and less later on. They accumulate almost no wealth by the normal age of retirement.

Risky assets should be very attractive to young households with many years to retirement, since human wealth will act as a substitute for the RFRR, allowing all financial assets to be moved into risky assets. The attractiveness of risky assets is reduced later in life as human wealth declines and financial assets accumulate. This is consistent with long-term perspectives on the risk-free asset, with the long bond or an inflation-indexed bond representing the risk-free instrument. Thus, investors at or near retirement will reduce risky asset allocations, and move to real bonds in an attempt to match near-term liabilities.

In summary, riskless / stable labor income creates a strong portfolio tilt toward risky financial assets. Variability in labor income can reduce that tilt, but not reverse it. Only if labor income shocks are highly variable and strongly positive in correlation to risky financial assets will an investor with labor income hold a more conservative portfolio than a retired investor with no labor income. Normally, labor income risk will only serve to generate a conservative portfolio at the limit that simulates a retiree's portfolio.

Gollier (2001) and Eeckhoudt, Gollier, and Schlesinger (2005) are two books are excellent reference treatises, providing tremendous amounts of information on expected utility, including many of the varied factors of the investor's opportunity set.[40] Much of the following analysis is taken from Kaufhold, working paper 2007:3. Major equations include of the dynamic process include the following. The overall goal is to maximize utility across periods. Power utility is normally used for U (z).

$$u(z) = \max_{\alpha_1} U(z, \alpha_1)$$

where alpha α_1 is the optimal strategy in the later period. The agent will maximize utility LT by:

$$v\ (z) = \max_{c0\ldots cs\text{-}1} \Sigma^{S\text{-}1}{}_{s=0}\, p_s\, u(c_s) \quad \text{s.t.}\ \ \Sigma^{S\text{-}1}{}_{s=0} \prod_s c_s = z$$

The degree of absolute risk tolerance LT is:

$$T_v\ (z) = -\ v'\ (z)\ /\ v''\ (z) = \ \Sigma^{S\text{-}1}{}_{s=0} \prod_s T\ (c^*{}_s)$$

The degree of absolute risk tolerance ST will be:

$$T_u\ (z) = -\ u'\ (z)\ /\ u''\ (z) = \ \Sigma^{S\text{-}1}{}_{s=0} \prod_s T\ (c^*{}_s)$$

Allocating ST risk over a lifetime of consumption has a time-diversification effect, making agents with LT time horizons more willing to take risk. (at 113). This effect can be stated as:

$$v\ (z) = \max_c \Sigma^{\,n\text{-}1}{}_{t=0}\, p_t\, u(c_t) \quad \text{s.t.}\ \Sigma^{\,n\text{-}1}{}_{t=0}\, c_t = z + n\ y$$

where, z is a given wealth accumulated prior to t = 0 date, pt is the discount factor associated with date t, and z + n y is the lifetime wealth. Optimal initial risk is determined by solving for $\max_{\alpha 0}$ Ev (z (α_0, x~).

When predictability of returns is brought into the analysis, LT horizon investors will take more risks early in life. The equations involve return through each period, $x{\sim}_t$. Predictability exists when $x{\sim}_1$ is correlated to $x{\sim}_0$. E $x{\sim}_0 > 0$, due to predictability of returns. The value function is:

$$u\ (z, x_0) = \max_\alpha E\ [\ (z + \alpha\ x{\sim}_1\ x{\sim}_1)^{(1-\gamma)} /((1-\gamma)\ \big|\ x_0].$$

The value function will be separable for each period, and the first period problem can now be solved via backward induction. Gollier (Opt. Port. Mgt, 2005) states the problem as:

$$v_{n-1}\ (z, s_{n-1}) = \max_{c1\ldots cs} \Sigma^{S}{}_{s=1}\, p_s\, u\ (c_s) \ \ \text{s.t.}\ \Sigma^{S}{}_{s=1} \prod_s (s_{n-1}) c_s = z$$

where the vector of prices in the last period depends upon states of nature s-1 that prevailed one period earlier. With CRRA > unity and mean reversion of returns, younger households should now have riskier portfolios. This is because the wealth effect dominates the precautionary effect, with overall marginal value of wealth increasing for riskier portfolios in the presence of predictability.

When the return distribution is not perfectly known, the optimal strategy is affected. This type of parameter uncertainty will tend to make the agent more conservative than with a return structure than is completely known to be predictable. This is consistent with

Barberis (1999) and others. In the early stages of learning of return distributions, it would therefore be prudent of the investor to be more conservative than at a later learning stage where more understanding exists as to return predictability. This was also noted in Gollier (Opt. Port. Mgt, 2005).

Jonathan Reiss (2006) conceptually follows Barberis to derive optimal allocations under uncertainty.[41] Reiss's opening comments are on point: "Merton (1973) (and Samuelson (1989)) noted that if returns are not independent over time, the optimal allocation varies with the investor's horizon. However, conventional practice has embraced the original results but has generally ignored the qualification."

The model with uncertainty is:

$$R_t = \mu e + \varepsilon m + \varepsilon r_t,$$

where μe is our estimate of the mean return ($\mu = \mu e + \varepsilon m_t$), and εm is a random variable with a mean of zero and standard deviation of σm, representing the error in our estimate.

The T-year variance is: $T\ \sigma r^{2} + T^{2}\ \sigma m^{2}$, where ρm represents the correlation in errors in the mean from period to period, with a standard deviation of σr. These optimizations maximize expected utility using the Constant Relative Risk Aversion (CRRA) utility function $U(W) = W\ (1- \gamma) – 1)/ (1 - \gamma)$ with $\gamma = 4$ (KCK: power utility).

Predictability with no ambiguity is introduced as:

$$R_t = \mu + b\ (d_t – D) + \varepsilon r_t,$$

where d_t is the dividend yield at time t, D is the "normal" dividend yield and b is the impact of dividend yield on return. The model with expectations risk added is:

$$R_t = \mu e + \varepsilon m_t + (be + \varepsilon b)\ X\ (d_t – D) + \varepsilon r_t,$$

where be is the estimated value of b, and εb is the error in that estimate with standard deviation σb. A graph (at p. 10) displays results.

Textbook note on Multi-period models.[42] The investor is assumed to maximize utility over lifetime consumption. The multi-period investment decision can still be reduced to the maximization of a one period model assuming: 1) consumer's tastes for goods and services are independent of future events (KCK note: this is the IID assumption); 2) consumers act as if the prices and consumption opportunities of the goods are known at the beginning of the period; 3) the consumer acts as if the distribution of one-period returns are known at the beginning of the period. Fama also showed that if the multi-period utility function exhibits more-to-less as well as risk aversion, then the derived one-period utility has the same properties as that period's consumption. Essentially, if the normal CAPM assumptions are used, then investors with a multi-period horizon would still use the standard CAPM. The zero-period CAPM may also be appropriate for multi-period investors. The particular single-period model that results depends on the additional assumptions being made.

Three models are noted in Elton and Gruber. The consumption-oriented CAPM uses a lifetime consumption maximization assumption. Returns on assets should be linearly related to the growth rate in aggregate consumption if the parameters of the linear relationship are constant over time. This is analogous to the simple form of CAPM, with the per capita growth rate in consumption replacing the rate of return on the market portfolio as the influence on the time series of returns and therefore, equilibrium returns. The model is:

$$R_{it} = \alpha_i + \beta_i C_i + e_{it}$$

Where, C is the growth rate in per capita consumption over time, and γ is the market price of consumption risk. $E(e_{it}) = 0$ by assumption, as is the covariance of e_{it} and $C_t = 0$. Beta is:

$$B_i = \mathrm{Cov}\,(R_{it}, C_t) / \mathrm{Var}\,(C_t)$$

The equilibrium return for any security is:

$$R_i = R_z + \gamma_1 B_i$$

These equations are similar to the zero beta CAPM, with return of the market portfolio replaced by the rate of growth in consumption between two points in time. Testing of the model has the same problems as with the zero beta CAPM, namely that the variable driving return (in this case per capita consumption) is difficult, due to sampling problems. Consumption estimates will contain sampling error; statistics are on expenditures, not consumption; expenditure are reported over a period of time, and not at one point in time; and monthly expenditures are available only after 1958. Elton, Gruber, (at 354) reviews possible solutions to these statistical problems. Breeden shows that the CAPM holds when growth in per capita consumption is replaced with the rate of return on a portfolio of assets that has maximum correlation with the appropriate consumption series. Referred to as a consumption portfolio, it contains returns from 13 industries, the T-Bill, LT government bonds, LT corporate bonds, and a junk bond premium. For the period 1929-1982, the correlation between a consumption portfolio and the CRSP value weighted index was 0.67. Tests with the consumption portfolio produced mixed results. Average return was linearly related to beta, and the intercept supported a riskless asset rather than a zero beta CAPM. The market price of risk, γ_1, was positive and statistically different from zero. Tests of efficiency however of both the consumption CAPM and the CRSP value weighted index were rejected however.

Another model involves inflation being added to the standard CAPM. Across time, equilibrium still exists, assuming constant risk aversion and a positive relation between inflation and market return. The market price of risk is higher than in the standard CAPM, however. Risk of any asset is not only the covariance with the market, but with the rate of inflation, as well. If the asset rate of return is positively correlated with the rate of inflation, the standard CAPM overstates the asset risk.

Merton (1973) developed a generalized inter-temporal CAPM. Referred to as the multi-beta CAPM, Merton maximizes lifetime consumption when faced with multiple sources of uncertainty over future prices, labor income, investment opportunities, etc. Investors will form portfolios to hedge away these risks. Future risk will affect expected returns on assets. The inflation model described above is a simple form of the multi-period CAPM. The multi-beta CAPM states that the expected return on any asset is related to the asset's sensitivity to a set of influences.

$$R_i - R_f = B_{im}(R_m - R_f) + B_{i1}(R_{i1} - R_f) + B_{i2}(R_{i2} - R_f) \ldots.$$

The model does not expressly state what these additional influences should be, or exactly how to hedge the risks that are represented in the model. Elton and Gruber states that the general concept of Merton's model is similar to APT, with its multi-factor sensitivities.

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[1] "Lifetime Portfolio Selection by Dynamic Stochastic Programming", *Review of Economic and Statistics,* vol. 51, no. 3 (August 1969): 247-257.

[2] Samuelson, "Risk and Uncertainty: A Fallacy of Large Numbers", (1963).

[3] Robert Merton, "Lifetime Portfolio Selection Under Uncertainty: The Continuous Time Case", *Review of Economics and Statistics,* vol. 51, no. 3 (1969), pp.247-257; and "Optimum Consumption and Portfolio Rules in a Continuous-time Model", *Journal of Economic Theory* 3 (1971): 373-413.

[4] Merton (1969), at 253.

[5] From Campbell and Viceira (2002), at 35.

[6] "What Practitioners Need to Know…about Time Diversification", Mark Kritzman, *Financial Analysts Journal,* vol. 50, no. 1 (January / February 1994): 297-323.

[7] Paul Samuelson, "The Long-Term Case For Equities And How It Can Be Oversold", *Journal of Portfolio Management,* vol. 21, no. 1 (Fall 1994): 15-24.

[8] Bodie, Kane, and Marcus (2003), at 211-212.

[9] See, Eugene Fama and Merton Miller, *The Theory of Finance* (1971); The multi-period model in the book was largely based on one of Fama's papers, "Multi-Period Consumption-Investment Decision," *American Economic* Review 60 (March 1970): 163-174.

[10] As noted by Jonathan Reiss in: "The Impact of Expected Return Uncertainty on Long-Term Risk and Investment Allocation Decisions", working paper #2, (2006), "Merton (1973) and Samuelson (1989) noted that if returns are not independent over time, the optimal allocation varies with the investor's horizon. However, conventional practice has embraced the original results but has generally ignored the qualification."

[11] See, Merton (1973). Comments on the 1973 paper come from: Cliff, Fin 6125 class notes, Fall 2007, Virginia Tech; as well as from direct quotes in the 1973 paper.

[12] Elton, Gruber et al (2003), at 218.

[13] In two important studies that measured relative risk preferences experienced by investors, both found decreasing absolute risk aversion (DARA), and conflicting results on relative risk aversion. See, Blume and Friend (1975); and Cohn, Lewellen, Lease, and Schlarbaum (1975).

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[14] See, Campbell and Viceira (2002), at 24-25, for an interesting observation that long run economic behavior suggests that relative risk aversion cannot depend strongly on wealth.

[15] Id., at 32.

[16] See, Samuelson (1969) and Merton (1969).

[17] From Gollier (2001????)

[18] See, Elton, Gruber, at al, at 245-246.

[19] The equations, examples, and analysis are from Stangeland and Turtle (1999).

[20] Stangeland and Turtle (1999), at 7-8 have a good discussion on determining threshold levels at which the effect that RRA has upon allocations can be reversed by RTM or MA.

[21] From Gollier (2001) and eeeckhoudt, Gollier, and Schlesinger (2005).

[22] The graph is taken from Cochrane (1999), who reuses a graph from Barberis (1999) to discuss return predictability and parameter uncertainty.

[23] Barberis, at 34.

[24] Jonathan Reiss, "The Impact of Expected Return Uncertainty on Long-Term Risk and Investment Allocation Decisions", working paper #2, (2006).

[25] The table is drawn from equations and conclusions reached in the Utility Chapter regarding power utility functions, and from two articles: Stangeland and Turtle (1999); and Kritzman and Rich (1998).

[26] E.S. Phelps, "The Accumulation of Risky Capital: A Sequential Utility Analysis", *Econometrica* 30 (1962): 729-743.

[27] "Risk and Uncertainty: A Fallacy of Large Numbers", Paul Samuelson, *Scientia,* vol. 57, no. 6 (April / May 1963): 1-6.

[28] "Proof that Properly Anticipated Prices Fluctuate Randomly", Paul Samuelson, *Industrial Management Review* 6 (Spring 1965).

[29] Paul Samuelson, "Lifetime Portfolio Selection by Dynamic Stochastic Programming", *Review of Economic and Statistics,* vol. 51, no. 3 (August 1969): 247-257.

[30] "Lifetime Portfolio Selection Under Uncertainty: The Continuous Time Case", Robert Merton, *Review of Economics and Statistics,* vol. 51, no. 3 (1969), pp.247-257.

[31] Eugene Fama and Merton Miller, *The Theory of Finance,* 1971.

[32] "Multi-Period Consumption-Investment Decision," Eugene Fama, *American Economic* Review 60 (March 1970): 163-174.

[33] Merton, Robert C., "An Intertemporal Capital Asset Pricing Model", *Econometrica* 41, (1973): 867- 887.

[34] (Fall 2007, Fin 6125, Prof. Cliff).

[35] Breeden, Douglas T., 1979, An intertemporal asset pricing model with stochastic consumption and investment opportunities, Journal of Financial Economics 7, 265-96.

[36] "Intertemporal Asset Pricing without Consumption Data", *The American Economic Review,* Vol. 83, No. 3 (June, 1993): 487-512.

[37] "Investing in the Long Run When Returns are Predictable", *The Journal of Finance,* 2000, no. 1, pp. 225-265.

[38] "Time Diversification: Fact or Fallacy", David Stangeland and Harry J. Turtle, *Journal of Financial Education,* Fall 1999, at 1-13.

[39] "Time Diversification: Fact or Fallacy", David Stangeland and Harry J. Turtle, *Journal of Financial Education,* Fall 1999, at 1-13.

[40] Christian Gollier (2001) *The Economics of Risk and Time*; Louis Eeckhoudt, Christian Gollier, and Harris Schlesinger, *Economic and Financial Decisions under Risk*, Princeton University Press (2005).

[41] The Impact of Expected Return Uncertainty on Long-Term Risk and Investment Allocation Decisions, working paper #2, (2006).

[42] From Elton, Gruber, et al, at 324.