General Equilibrium Models For Use in Asset Pricing

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Abstract: This paper examines various models that may be of use in incorporating macro factors into asset pricing models. The focus is on theoretical considerations linking the macro and micro environments.

Introduction – Data in Search of a Theory

This is another working paper in a continuing series that discusses empirical and theoretical aspects of asset pricing. In the first paper of this series, Kaufhold 2008:1, the CAPM and the three-factor Fama and French models were reviewed. Data was developed on both the original 366 month period used by Fama and French as well as the more comprehensive period of 522 months through 2006. Regressions conducted by the author duplicated Fama and French's initial findings as to the significance of not only the market, but size and value factors, as well. The more expansive period also produced positive findings on all three F&F factors.

In 2008:2, Carhart's fourth factor of momentum was considered. When momentum was used as an additional factor, both the 366 month and the 522 month data produced significant results over the CAPM and the three-factor model. Reversionary variables also generated favorable results when used as a fourth factor.

In 2008:3 and 2008:4, intertemporal models and macro variables were studied. A literature review showed that various macro factors produced positive results when used in the F&F multi-factor format. Some of these variables include GDP growth and expectations of growth, unexpected inflation, interest rates, and business-related defaults.

Empirically, multi-variate regression equations have typically been used to deduce the macro factors. The following equation from Aretz, Bartram, and Pope (2005) is illustrative of such efforts:

$$R^{E}_{t-1,tp} = \beta_{0p} + \beta_{1p} MYP_{t,t+12} + \beta_{2p} UI_{t-1,t} + \beta_{3p} DSV_{t-1,t} + \beta_{4p} ATS_{t-1,t}$$
$$+ \beta_{5p} STS_{t-1,t} + \beta_{6p} FX_{t-1,t} + \beta_{7p} OIL_{t-1,t} + \varepsilon_{t-1,tp}$$

Where MY Pt,t+12 is the change in expectations of one year ahead industrial production growth over month t, UIt−1,t is unexpected inflation in month t, DSVt−1,t is the change in the aggregate survival probability in month t, ATSt−1,t and STSt−1,t are changes over month t in the average level and the slope of the term structure. FXt−1,t is the change in a multilateral U.S. dollar exchange rate and OILt−1,t represents the change in a raw materials price index.

The Aretz, Bartram and Pope paper as well as many other works have shown that macro factors placed into multi-factor models are relevant and statistically valid. Theoretical understanding of the reasons why macro variables could work in an asset pricing regression could use far greater analysis, however. F&F even describe their modeling efforts as being "ad hoc" in nature.

Asset pricing equations typically start with consideration of dividend of the dividend stream stated in present value terms:

$$P_1 = D_0 (1 + g) / r; \text{ and then:}$$
$$P_1 = D_0 (1 + g) / (k-g).$$

Where P is price, D is the dividend stream; g is the dividend growth rate; r is the discount rate; and k is the capitalization rate or required rate of return. Equations using fundamental analysis highlight the difficulty with adding macro variables into the process. The following equation was noted in Aretz, Bartram, and Pope (2005):

$$MV_0 = B_0 + \sum_{t=1}^{\infty} [E_0^q RI_t / (1 + r_t)]$$

Where, MV_0 is the market value of an asset; B_0 is the book value of the asset; $E_0^q (RI_t)$ is the probability weighted residual income of the asset; and $(1+r_t)^t$ is the discount factor. With this equation, market value becomes a function of book value, residual income, and the term structure of interest rates. Adding price of risk (PR_0) to the equation generates:

$$MV_0 = B_0 + \sum_{t=1}^{\infty} [E_0^q RI_t / (1 + r_t)] - PR_0$$

Market value of an asset is now dependent upon a fundamental factor, book value, the asset's exposure to term structure risk, and other uncertain risk factors affecting future cash flows. This last factor could be influenced by unstated macroeconomic variables in addition to the term structure of interest rates. Using equations that are typically associated with firm-level understanding of economic activity impliedly point towards the use of macro variables, but leave out numerous mechanisms that are obviously at work in the macro environment.

Theory In Search of a Linkage

Many efforts have been undertaken to develop a theoretical understanding of both micro and macro level risk factors in asset pricing. The general goal has been to unite or combine macro-level economies with capital asset pricing models. Initially, consumption based models were used to incorporate some macro factors. More recently, dynamic general equilibrium models and production-based asset pricing models have been used to establish a central theory explaining both macro and micro effects upon asset pricing.

Merton (1973) was the first person to make the connection between macro and micro processes when he added interest rates into the Capital Asset Pricing Model.[1] Merton's Intertemporal Capital Asset Pricing Model (ICAPM) postulated that the investor optimizes risk and return within the traditional Markowitz formulation as well as engaging in an inter-temporal hedge against uncertain future events (such as interest rate fluctuations) that may affect the opportunity set. The basic ICAPM equation is:

$$\alpha_i = r_f + \beta^m_i (\alpha_m - r_f) + B^h_i (\alpha_h - r_f)$$

$\beta^m_i (\alpha_m - r_f)$ is the market component, while $B^h_i (\alpha_h - r_f)$ is the hedging component in the above equation. The analysis can be generalized to include hedging against any uncertain macro event occurring in the future. In fact, the standard form of the CAPM is now considered by some to be a special case of the ICAPM, limited to a non-temporal investment universe with no hedging component for future events. Fama and French (1996) also believe that their multi-factor model takes on aspects of the ICAPM, just with value and size factors being used as the additional items influencing asset pricing beyond the market factor. The Arbitrage Pricing Theory's multi-factor process also takes on many aspects of the ICAPM. Indeed, if the forward hedging component is added to the APT, it becomes almost indistinguishable with Merton's multi-beta process.

With Merton's methods, utility can be estimated over lifetime consumption, and can be thought of as current utility plus an indirect utility function representing the optimized value of all future utility. The model can then be used to minimize variation in consumption growth. The indirect utility function, J(W), is stated as:

$$J(W, t, X) = \max Et [\int_t^T U [C(s), s] ds + B (W (T) T)]$$

In the ICAPM, agents set the marginal utility of wealth equal to the marginal utility of consumption along the optimal consumption path. The envelope condition then becomes U(C) = J(W).

Thoughts on consumption became more prominent when Lucas (1978) analyzed a one-good, pure exchange economy with identical consumers. Lucas equated the marginal rate of substitution of current to future consumption to the market rate of return on securities. In micro-economic settings, the MRS is often viewed as being equal to market pricing of goods. It is therefore not that difficult to theoretically equate the rate of substitution to asset pricing returns.

[1] Some of the comments regarding Merton and Breeden are taken from a VA PHD teaching outline, Cliff (2007).

Breeden (1979) then extended Merton and Lucas into a full asset pricing model incorporating aggregate consumption, described as the consumption-based asset pricing model (CCAPM). This model treats consumption growth as the single state variable, and then hedges against changes in consumption. The standard CAPM also has this same consumption interpretation since it is essentially a single period model with end-of-period wealth that is tantamount to consumption. Breeden then minimized the variance in consumption, rather than wealth. Where data on consumption is available, consumption thus serves as the single variable driving the returns process. When it is not available, additional state variables are then included in a multi-factor format ala Fama and French. The expected return on any risky asset income stream is stated across time periods as:

$$E(r_{t+1}) = (r_f^t – [1 / (1+r_t^f)]) \text{ cov } [\beta \text{ U}' [(c_{t+1}) / U'(c_t), r_{t+1}]$$

With this equation, expected return over two periods now becomes a function of the utility of current consumption, utility of future consumption, the risk-free rate of return, and the discount factor. The stochastic return, r (s) can then be described as:

$$\sum \pi (s) [\beta \text{ U}' (c(s) / U'(c)] (1 +r(s)) = 1$$

Wickens (2008) incorporates Breeden's thoughts on the CCAPM into a two-period dynamic general equilibrium (DGE) model. He showed that the same process can explain both the optimal consumption path as well as asset pricing via the CCAPM. Starting with the household consumption problem of:

$$V_t = \sum B^s U (c_{t+s})$$
$$\text{s.t. } \Delta a_{t+1} + c_t = x_t + r_t a_t$$

where x_t is income and a_t is the real stock of assets, uncertainty can be built in as expectations, or: $V_t = \sum B^s E_t [U (c_{t+s})]$. As a general proposition, utility can be stated as:

$$V_t = G (U (c_t), E_t (V_{t+1}))$$
$$\text{And, } V_t = U (c_t) + \beta E_t (V_{t+1})$$
$$\text{Then, } V_t = \sum^T_{i=0} \beta^i E_t [U (c_{t+i})] + \beta^T E_t (B (a_T)]$$

The last equation uses a finite horizon T. The CAPM is the second part of the above-equation, $\beta^T E_t (B (a_T)]$. Thus, the CAPM represents the immediate value of assets without considering future consumption. General equilibrium occurs at the optimization of current consumption via the CAPM and the present value of future consumption. The first-order condition is:

$$\partial V_t / \partial c_t = \partial U_t / \partial c_t - \beta E_t (\partial V_{t+1} / \partial c_t) = 0$$

The discount factor can be inserted into the FOC by:

$$\partial V_t / \partial c_t = \partial U_t / \partial c_t - \beta E_t [(\partial U_{t+1} / \partial c_{t+1})*(1 + r_{t+1})] = 0$$

Expectations can be derived from the FOC as:

$$E_t\,[(\beta\, U'_{t+1} / U'_t)*(1 + r_{t+1})] = 1$$

This becomes the equivalent of the asset pricing equation of the CCAPM, above. Starting with an aggregate household consumption problem typical of macro-economic models, the equivalence with the CCAPM is derived. With these last few equations, macro considerations of consumption growth and inflation are able to impact asset pricing. In an efficient market, risky assets are priced from the risk free asset plus risk premiums that reflect macroeconomic sources of risk.

The stochastic discount factor of a consumption-based asset pricing model can be expressed as:

$$E_t\,[M_{t+1}\,(1 + r_{t+1})] = 1$$
$$\text{And, } M_{t+1} = (\beta\, U'_{t+1} / U'_t)$$

To Wickens, multi-factor asset models, such as Fama and French, reflect various stochastic discount factors. The CAPM assumes a single discount for pricing volatility of the market as a whole. The CCAPM assumes a sole discount for consumption growth. In multi-factor models, the factors are typically chosen not on the basis on intertemporal marginal rate of substitution (as was the case with Merton), but selected based on available data.

In the asset pricing models, the risk free rate is often used for the discount rate. With the models under discussion however, the marginal rate of substitution in consumption between two consecutive periods, M_{t+1}, is used as the discount. The relation between the two types of discounts is:

$$E_t\, M_{t+1} = 1 / (1+r_t^f).$$

Through this discount equivalency, general equilibrium can be achieved in both macro and micro level environments.

Cochrane (2006) concentrated on consumption-based models with first-order conditions and production analysis. Asset returns could be linked to the utility of consumption across two periods:

$$m_{t+1} = \beta\, \mu_0\,(c_{t+1}) / \mu_0\,(c_t)$$

where m now is the contingent claims price.

Reffett and Schorfheide (2000) begin with a household model economy. The household attempts to maximize expected lifetime discounted utility through consumption and savings decisions, while a firm sells goods and rents labor from households. Constant returns to scale are assumed with:

$$f_t (K_t, N_t, X_t) = T_t K_t^{\alpha} (N_t X_t)^{\alpha}$$

With K being the capital stock at time t; N being labor; X is the labor augmented technological process. T is the total factor productivity which evolves according to a AR(1) process. The factors of production can be converted to consumption, C_t, and Investment, I_t, such that:

$$[C_t^{\zeta} + \theta_t I_t^{\zeta}]^{1/\zeta} = f_t (K_t, N_t, X_t)$$

Similarly to above-outlined models, the stochastic discount factor is:

$$1 = E_t [M_{t+1} R_{t+1}]$$

Where R is the asset return.

Business cycle models have also been utilized. Tallarini (2000) uses Epstein-Zin utility in a standard real business cycle model:

$$U_t = \log C_t + \theta \text{ Log } L_t + \beta / \sigma \log [E_t (e^{\sigma U t+1})]$$

Due to the nature of the Epstein-Zin recursive utility function, risk aversion can be changed in this model without affecting the intertemporal substitution. As noted in Campbell and Viceira (2002), this feature of Epstein-Zin allows the link between risk aversion and inter-temporal substitution to be severed while still retaining scale independence of power utility.

Production based models have also come into vogue. Belo (2005) uses a production model to discuss pricing returns, generating a linear macro-factor process:

$$m_{t+1} = 1 + \sum_i b_i \Delta y_{i, t+1}$$

Where the contingent claims price, m, becomes a function of output, y. Regressions generate large values for b_i. Thus, the beta coefficient of output in a production model has similar results as the independent factors of Fama and French modeling, just with output being the independent variables rather than value and size return factors.

Conclusions

Empirically, size, value, momentum, reversion, and various macro factors generate statistically significant findings in multi-variate regression analysis on asset pricing. Developing a theoretical basis for such positive findings has been a challenge for the investment community, however. Both consumption and production based models have shown promise in explaining the statistical relevance of multi-factor asset models, although the discussion is far from mature.

Marginal utility can be expressed in terms of the stochastic discount factor, $M_{t+1} = (\beta \, U'_{t+1} / U'_t)$. This provides the critical link between the micro investment world and the macro universe. With consumption based models, asset pricing becomes dependent upon the optimal consumption growth path across multiple time periods. With production based models, pricing is partly a function of aggregate output. With either model, dividend payout is still the basis of asset valuation. Arguably, the only "state" variable remains the firm dividend stream, discounted to present value. Multi-factor asset models may be merely reflecting numerous micro and macro variables that ultimately affect the present cash value of the future dividend stream through a discount rate equivalency.

The specific mechanisms by which aggregate consumption and production cause dividend and cash flow fluctuation needs more analysis, however. In many ways, the real inquiry shifts to the specific processes at work.

To simply state that the stochastic discount factor is the critical link in determining macro and micro general equilibrium merely creates a series of additional questions. What exact micro factors are impacted by specific macro events? And by what means? And are there micro impacts simultaneously occurring to many firms from any of several macro factors? Is this the real source of asset pricing correlation? A good portion of portfolio theory, let alone micro firm theory, then begins to be subsumed within a macro climate. In the end, asset pricing theory may simply be incomplete without an extensive analysis of various macro factors which impact firm-level matters.

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