A Comparative Analysis of the
Special and General Cases of Utility Maximization

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Kevin Kaufhold
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1. Introduction

This article explores the special and general cases of utility maximization for use in portfolio analysis and management. It is meant to supplement and complete earlier works by the author (Kaufhold, 2007; 2008) that were necessarily limited to survey and introductory levels. This paper is thus designed to provide the mathematical reasoning and logic behind the development of portfolio theory based on expected utility.

The effort starts out with a short review of the development of the special case of utility as applied to the investment decision-making process. A critique of the special case is then made. In some instances, the usage of the quadratic can be defended, largely by continued reliance upon normality of diversified portfolios as well as by extending means-variance procedures to include liability streams. The limited nature of the special case ultimately leads however to a more comprehensive analysis of the general case of utility. Static and dynamic methods are explored. The effect of return predictability is also examined. The article concludes with a discussion of diversification and allocation decisions within the general position of expected utility.

2. The Special Case of Utility Maximization

To appreciate the dexterity and usefulness of expected utility, a review should initially be made of the special situation when returns are normally distributed. The analysis is kept brief, since the special case has been extensively discussed in any number of texts. More thought is then given to examining the limitations and particular issues surrounding the special case.

2 (A). Basic Review of Special Case

Mathematically, utility of wealth is a function of wealth, or $u(w) \sim \int(w)$, where utility of wealth, u (w), is related in some fashion to the wealth itself, defined as w. Beginning with

Bernoulli (1738) many hundreds of years ago, utility of wealth was seen as being potentially non-linear in nature, with utility increasing as wealth increased, but perhaps at a declining rate. von Neumann and Morgenstern (1948) developed certain assumptions or axioms, providing the basis for expected utility, whereby economic decisions could be evaluated under conditions of uncertainty. Expected utility can be viewed as the probability weighted utility decision of an agent:

$$E(u) = \sum_w \; p(w)\, u\,(w)$$

Utility can be maximized as:

$$\max E_t\, u\,(w_{t+1}) \text{ s.t. } w_{t+1} = (1 + R_{p,\, t+1})\, w_t$$

where, E_t is the expected value in time t; u (w_{t+1}) is a utility function; and R_p is the return distribution of a portfolio of assets. When the return distribution is assumed to be normal, or approximately so, only the first two moments of the distribution then matter, that of mean and variance. In this special situation, the utility function, u (w_{t+1}), can be expressed as a quadratic equation.

$$u\,(w) = a\,w - \tfrac{1}{2}\, w^2, \text{ for } w <= a$$
$$\text{More generally, } u\,(w) = aw - bw^2$$

Where, E(w) is expected wealth. Further derivation of the equation leads to:

$$\sigma^2_w = E(w - E(w))^2.$$
$$E\,(u\,(w)) = E(w) - b\,[\,\sigma_w^2 + (E\,(w))^2\,]$$
$$\text{more simply as: } u = E(r) - \lambda\, \sigma^2$$

with λ being the risk-aversion coefficient. The quadratic utility function can be expanded to the familiar two-asset variance equation first developed by Markowitz (1952, 1959):

$$\sigma^2_p = a^2\, \sigma^2_x + b^2\, \sigma^2_y + 2\, a\, b\, \sigma_{xy}.$$
$$\text{and, } \sigma_{xy} = \rho_{xy}\, \sigma_x\, \sigma_y.$$
$$\text{then, } \sigma_p = \sqrt{\sigma^2_p}$$

where, σ_{xy} is the covariance of the two assets, and with ρ_{xy} being the correlation between assets x and y. Since the quadratic is the only utility function capable of being expressed in direct pricing terms, is has become the standard equations among many portfolio theorists, from Markowitz forward. All other utility functions are solved in "utils", which is rather esoteric to everyone concerned, investors and professionals alike.

The rate of return on the portfolio is a weighted average of the expected returns of the component assets: $E\,(\,r_p\,) = w_b\, E\,(\,r_b\,) + w_s\, E\,(\,r_s\,)$. The weights sum to 1 and are positive, or: $w_s + w_b = 1.0;\ 1.0 <= w_s => 0;\ 1.0 <= w_b => 0.$

The 1st derivative of the above equation generates the minimum variance on the portfolio:

$$w^* = (\sigma^2_2 - \sigma_{12}) / (\sigma^2_1 + \sigma^2_2 - 2\,\sigma_{12}).$$

Sharper, Lintner, and Mossin independently developed the Capital Asset Pricing Model (CAPM) between 1964 and 1966. The general form of asset pricing is a simple linear regression equation, $y = a + bx$, with the y intercept being the risk free rate, r_f, and Beta, β, being the slope of the independent variable, x. The security and capital market lines lie at the heart of the CAPM, representing the risk-return off of securities and capital assets. The relevant equation for the SML is:

$$E\,(r_i) = r_f + \beta i\,[\,E\,(r_m) - r_f\,]$$

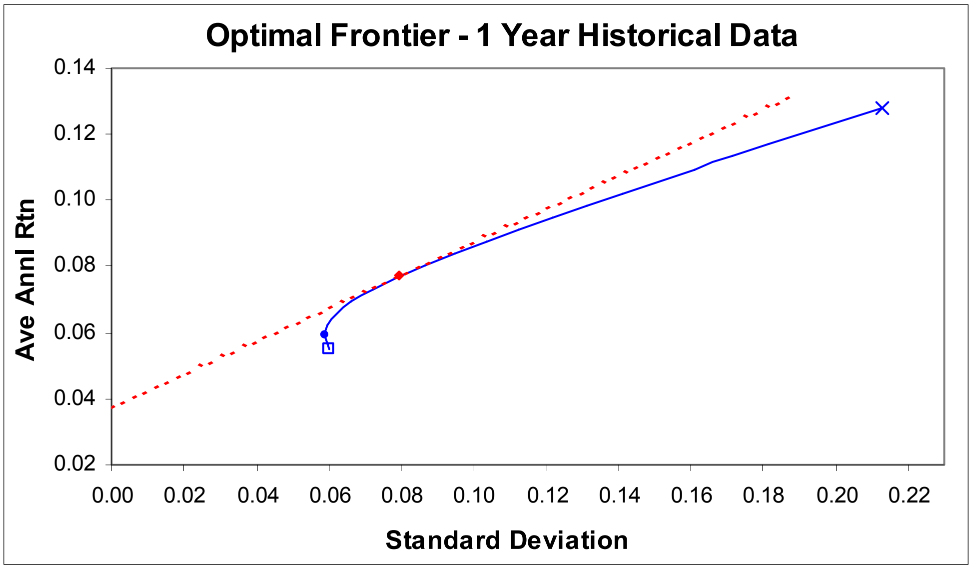
Where, $E(r_i)$ is the expected return to the ith asset; and r_m is the return to the market variable. Portfolio beta is the weighted sums of all individual betas of the portfolio: $\beta_p = \Sigma\, wi\, \beta i$

For a two-asset portfolio, Markowitz's efficient frontier can be combined with the market line of the CAPM to generate an optimal equilibrium between risk and return of the two assets.

$$w_b^* = [(E\,(r_b) - r_f)\,\sigma^2_s - (E\,(r_s) - r_f\,)\,\sigma_s\,\sigma_b\,\rho_{sb}]\,/$$
$$[(E\,(r_b) - r_f\,)\,\sigma^2_s + (E\,(r_s) - r_f\,)\,\sigma^2_b - (E\,r_b) - r_f\,+ E\,(r_s) - r_f\,)\,\sigma_s\,\sigma_b\,\rho_{sb}]$$
$$w_s^* = 1 - w_b^*$$
$$r_p^* = r_s\,w_s^* + r_B\,w_b^*$$
$$\sigma^2{}_p{}^* = w^2{}_s{}^*\,\sigma^2_s + w^2{}_b{}^*\,\sigma^2_b + 2\,w_s^*\,w_b^*\,\sigma_s\,\sigma_b\,\rho_{sb}$$

The portfolio variance for K assets becomes: $\sigma^2_p = \sum w_i^2 \sigma^2_i + \sum\sum wi\, wj\, \sigma_{ij}$,. Because the covariances of assets can take on overwhelming proportions, a portfolio containing a large number of securities can be reduced to the sum of the weighted covariances. For a portfolio of many assets, the variance then reduces to: $\sigma^2_p = \sum\sum wi\, wj\, \sigma_{ij}$,

Optimization can be expressed visually, as shown in the following graph. Optimality occurs where the security market line becomes tangent to the efficient frontier. A 100% bond position is noted as a box in the following graph; the minimum variance point, a blue dot; a 100% equity position as a blue cross; and the point of optimality is the red dot.



When returns are assumed to be independently distributed, means-variance generates a fully diversified holding of assets.

$$a^*_1 = (1 / A) [(\mu - x_0) / \sigma_{ii})]$$

where, a* is a vector of optimal shares invested in the risky assets, and A is the degree of absolute risk aversion. Risk aversion allows the agent to substitute the risk-free asset into the optimized portfolio. The optimal allocation of risky assets then simply becomes a function of the variance of the return distribution ($\mu - x_0$) divided by the co-variance (σ_{ii}) of the portfolio assets.

2 (B). Particular Issues

Many of the assumptions of the MPT can be relaxed without major disruptions occurring to the basic model. Short sales and leveraging can be easily constrained by Lagrangian equations so that long-only portfolio positions can be achieved. The RFRR can be changed to commercial lending rates. This would generate a kinked market line, but otherwise would not affect the model's validity. Input estimation uncertainty would potentially widen the frontier and the CML from definitive lines and curves to ranges based on probabilities. This too would not change the overall usefulness of the special case, as a market portfolio optimum would merely develop into a range of likely allocations. Adding transaction and research costs would reduce the expected rate of return and may also generate some inefficiencies, due to a hurdle rate now being necessary to remove mispriced assets. Heterogeneous investors would also create probable CML and SML bands, due to varying expectations. The introduction of taxes would lower the slope of the market lines by lowering the reward to risk ratios. Adding non-marketable assets could also lower the return-risk trade-off. Inflation would increase the overall price of risk.

While relaxing these assumptions does not have an adverse impact on the MPT, when a few other critical assumptions are changed, the vitality of the special case erodes. The following items are particularly noteworthy.

2 (B) (1). The Normality Assumption

By using stochastic dominance procedures where no utility functions are used, a comparison can be made against a means-variance investment universe. Generally following Eeckhoudt, Gollier, and Schlesinger (2005), two risk free assets with the same return structure will have the same utility of wealth. Stated mathematically: $u\ (w_2) = u\ (w_1)$. When risk, or variance of return, is introduced to the second asset while the first asset has the same return with no risk, there is a clear preference for the risk-free asset: $Eu\ (w_2 + z\sim) <= u\ (w_1)$, where, Eu is expected utility and z~ is a zero-mean risk with an expected pay-off of zero, or $Ez\sim = 0$.

Once variance of return is introduced into both assets, the preference for one or the other asset becomes more complicated. Where both assets have the same expected return, but asset two has a greater variance then asset one, a clear preference still exists for the first asset: $Eu\ (w_2 + z\sim) <= Eu\ (w_1 + z\sim)$. This is an example of a mean-preserving spread.

But in the event that both assets have the same variance and the second asset now has a smaller expected return, the entire probability distribution has shifted downward for the same level of risk. This is considered first-degree stochastic (or random) dominance (FSD) deterioration:

$$Eu\ (w_2\sim) - Eu\ (w_1\sim) = - \int_a^b u'\ (w)\ ((F_2\ (w) - F_1\ (w))\ dw$$

Where, F_i (w) is a continuous cumulative probability distribution. A second-degree Stochastic Dominance (SSD) deterioration then occurs when any form of risk is added to a FSD deterioration. Thus:

$$Eu\ (w_3 + z\sim) <= Eu\ (w_2 + z\sim).$$

As Elton, Gruber, Brown, and Goetzmann, (2003) notes, stochastic dominance procedures will still lead to the same portfolio allocations as with MVO. A FSD downward shift implies the existence of the entire frontier so long as returns are normally distributed. SSD is consistent with that part of the frontier above the minimum variance point. Even with the quadratic's limitations regarding normality, means-variance will generate the same optimal set of allocations as do first and second order dominance positions.

Once skew is introduced into the probability distribution however, two assets having the same SSD characteristics may be indeterminate. MPT cannot ascertain which of two assets on the frontier will be preferred, given the same FSD and SSD conditions. Since returns of many individual assets are lognormally distributed with a skew being present, this potentially is a major concern.

This difficulty is largely offset however by the realization that a broadly diversified and balanced portfolio generates an approximately normal distribution of returns. For instance, Fama (1976) found that the quarterly and yearly returns of diversified equity portfolios are closer to normal than are daily and weekly returns. Arguably, the loss of normality in the return structure of individual assets is not of great concern so long as a portfolio is fully developed and mature.

2 (B) (2). Externally Driven Liabilities

For balanced and diversified portfolios, the special case of utility maximization may not be severely damaged by downside risk that is typical of a lognormal skew in individual returns. With MVO and CAPM basically being limited to asset-only determinations however, the special case can become rather strained when externally generated liabilities are introduced into the investment process. This is a very important limitation, since liabilities are integral components of virtually all professional-level investment policies.

A.D Roy (1952) was likely the first person to consider liabilities within a portfolio context. Roy developed a Safety First Criterion, whereby the optimal portfolio should minimize the probability that portfolio return will fall below a designated threshold level.

$$SFR = \min p\,(r_p < r_L)$$
$$SFR = [\,E(r_p) - r_L] / \sigma_p$$
$$p\,(r_p < r_L) \text{ is: } N\,(-\,SFR).$$

Where, SFR is the Safety First Ratio. The probability that the return will be less than the SFR can be ascertained by simply referring to a standard normal cumulative density function.

Currently, both Asset-Liability Modeling (ALM) and Monte Carlo simulations are extensively used to bring liabilities within an asset-based means-variance analysis. ALM is an accounting rules-based process while Monte Carlo is more mathematical in nature. As noted by Cable and Mike Rust (2005) and Ruloff (2006), ALM uses accounting type of equations such as:

Minimum Funding Requirements = NC + UAL – Credit Balance
Credit Balance = total cumulative C – minimum required C
C = Contributions; NC = normal cost
Unfunded Actual Liability (UAL) = Actuarial L + Actuarial A

The overall objective of ALM is to ensure the full funding of the forward liability stream by at all times maintaining sufficient portfolio assets. Notice how much the effort has moved away from an evaluation of only the means and variance of the asset return distribution.

Monte Carlo is analytically closer to MVO techniques, since optimized portfolio risk and return levels are used to develop a probability of portfolio depletion resulting from the expected liability stream. Monte Carlo simulations can be developed by:

$$wt = (1+rt)(wt-1 - awt-1 + (awt-1 * p))$$

where, w_t is wealth at year-end t; r_t is the total return in period t; aw is the annual withdrawal, and p is the inflation rate or COLA used (if any). r_t is randomly generated within a lognormal probability distribution whose mean and standard deviation is determined by the assumptions of the model. As Ibbotson (2005) states, a lognormal distribution is used in many simulations because a normal distribution could overestimate extreme values with a potential for infinite gains and losses. Ibbotson (2005) details further procedures and assumptions for Monte Carlo modeling.

While the traditional Markowitz formulation leaves out any consideration of liabilities, both ALM and econometric endeavors can supplement standard portfolio theory, albeit in a somewhat extended fashion from the original theoretical underpinnings.

2 (B) (3). Time Horizons

So far in the critique, MPT has survived relatively intact, taking refuge in the normality of fully diversified portfolios, and being supplemented by liability-based models. But now we turn to a vexing problem. The standard treatment of portfolio theory assumes a unitary time horizon. This is rather perplexing to many investors and professionals who are used to long-horizon financial planning.

Early theoretical efforts gave support to a unitary time horizon applicable to homogenous investors. Using a discrete time model, Samuelson (1969) demonstrated that with an iso-elastic marginal utility function, $U'(C) = C^{\gamma-1}, \gamma < 1,$ the optimal portfolio decision is independent of all consumption savings decisions. This leads to a constant percentage allocation. Merton (1969) showed that that the optimal portfolio rule for an infinite horizon model with constant relative risk aversion (CRRA) and log utility of $U(C) = \log C$, and $U(C) = = C^{\gamma} / \gamma$, will be independent of the consumption decision. Merton also looked at CARA utility where: $U(C) = - e^{-\eta C} / \eta, \eta > 0$. With this function, consumption is no longer a constant proportion of wealth (as is the case with log utility), but it is still linear with wealth.

More recently, Campbell and Viceira (2002) noted that a time invariant result occurs with log utility even where asset returns are serially correlated. An investor maximizes expected log returns as: $r_{pK, t+K} = r_{p, t+1} + \ldots + r_{p, t+K.}$ Long-horizon K-period log returns will merely be the sum of one-period log returns. This sum will be maximized by choosing in each period the allocation that is optimal for a single-period log investor.

The time irrelevance view is premised upon very precise and carefully crafted assumptions, namely log utility or power utility with CRRA and independent returns. When power utility is used with varying relative risk aversion and varying return distribution structures, optimal asset allocations could increase, stay the same, or decrease over time. Power utility was modeled in Stangeland and Turtle (1999) as:

$$u(z) = (1 / (b-1)) (a + b z)^{(1 - 1/b)}$$

where, z is wealth, and u (z) is the utility of the wealth z. Also, $b>0$; and $z > [-a / b, 0]$. Absolute risk aversion is determined by: $A(z) = -u''(z) / u'(z) = 1 / (a + b z)$. Relative risk aversion is then calculated as: $R(z) = -u''(z) z / u'(z) = z / (a + b z)$. Changes in relative risk aversion are accomplished through the value of the "a" parameter. In the one instance of constant relative risk and independent returns, time is indeed irrelevant to the allocation decision. But in any other instance, allocations would change across time. With decreasing relative risk aversion and return predictability, for instance, allocations will increase across time. With increasing risk aversion and mean acceleration, equity allocations will decrease across time. When decreasing relative risk exists with mean acceleration, allocations will be come dependent upon the strength of these contradictory economic effects.

Similar results were reached in Barberis (2000), who showed that the introduction of return predictability greatly increased equity allocations in the case of power utility and CRRA assumptions. The power equation employed was: $(W) = W^{1-A} / (1 - A)$, where W is wealth, and A is the risk aversion coefficient. Dynamic rebalancing occurs with: $\max_{to} E_{to} (W^{1-A}_{K} / (1-A))$. Barberis then added parameter uncertainty in returns, with: $P (R_{T+T^{\wedge}} | r)$, where $r = (r_1 \ldots r_T)$. For predictable returns and CCRA, allocations increase to 100% equities by the ten-year hold. But when uncertainty is introduced, optimal equity allocations peak at around 60%, and then decline somewhat in the very long holding periods, using certain risk aversion settings.

Thus, a unitary time horizon assumption per MPT can only be supported in the very limited situations of either: 1) log utility; or 2) power utility with CRRA and independent return (IID) assumptions. With any other assumption or utility function, time is an important factor in determining appropriate portfolio allocations. This conclusion is completely missed when the analysis is limited to only means-variance procedures.

2 (B) (4). Risk Aversion

Another troubling aspect of MPT is the assumption of constant risk aversion. Since the quadratic equation is inherently increasing in both absolute and relative risk aversion, an immediate inquiry comes to mind as to the appropriateness of using a quadratic for investment modeling purposes.

Gollier (2001) engages in an excellent analysis of the general topic by extensively detailing risk aversion concepts. Risk averse agents will have a utility function that is concave, with $E(w) <= u(w)$. Most investors are thought to be risk averse, and empirical evidence tends to support the proposition. Different agents will have differing amounts of risk aversion, however. Absolute risk aversion (ARA) measures the degree of concavity in utility. ARA is measured in units of wealth, as varying monetary amounts will be necessary to induce people to take risk. A (w) measures the maximum amount that an agent will pay to get rid of a small risk. The applicable equation is: $A(w) = - u''(w) / u'(w)$.

Relative risk aversion (RRA) is simply the percentage at which marginal utility decreases for a percentage decrease in wealth. Relative risk aversion is noted as: $R(w) = - w\, u''(w) / u'(w) = w\, A(w)$. While there is a general consensus that investors are risk averse and also are decreasing in their absolute risk aversion, there is far less agreement on whether investors are decreasing in their relative risk. The empirical evidence is contradictory on whether relative risk is increasing or decreasing.

The inherent nature of the quadratic to display increasing risk aversion proves to be most difficult for portfolio analysis. Gollier (2005) feels that risk premiums for additive risks are likely to decrease with increasing wealth, rather than increase with accumulating risks as is the case with the quadratic function. Gollier additionally notes that there is no obvious reason for the quadratic to represent the attitude towards risk of agents in the real world. Elton, Gruber, et al (2003) acknowledges that the MPT assumption of CRRA largely stems from convenience rather than descriptive accuracy. It is quite possible that different investors, or the same investors at different periods in their lives, will have differing preferences on risk aversion. The lack of consensus on what type of relative risk investors possess may be indicative of shifting and differing risk aversion preferences among investors.

If an investor or financial planner is not overly concerned with the nuances and finer points of utility, then using MVO may be sufficient for many applications. To set forth a full and capable theory explaining the investment process however, utility functions should be deployed which more adroitly model investor behavior.

3. The General Case of Utility Maximization

Given the limitations of the quadratic equation and certain critical assumptions of the MPT, many utility theorists long ago abandoned the usage of the quadratic for investment modeling in favor of the power series of utility functions. It is therefore very important to have a solid understanding of expected utility (EU), as generally stated. The following analysis is taken from several sources, most importantly Eeckhoudt, et al (2005); Gollier (2001); and Elton, Gruber, et al (2003).

3 (A). Review of the General Case

Since the general case of utility is not as commonly known as the special case, the basic model will be more thoroughly detailed than the cursory examination which was provided to the special case in Section 2(A), above.

3 (A) (1). Static Methods

The analysis begins with the Arrow-Debreu economy, named after seminal works by Arrow (1953) and Debreu (1959). The standard portfolio problem can be expressed as:

$$\max_{\alpha} Eu\,(\alpha\, x{\sim}_1 + (w - \alpha)\, x{\sim}_2\,)$$

Where, α is the amount invested in the first risky asset, $x{\sim}_1$, and w is wealth. This equation becomes a base-line for the model, with more complexities being added through revisions to the equation. Optimal risk exposure is established by:

$$\max_{c0\ldots..cs} \Sigma^{S}_{s=1}\, p_s\, u\,(c_s)$$

The probability-weighted utility of consumption, (c_s), is now maximized for each type or claim of consumption. An indifference curve can be sketched out between two risky choices, using the equation of: $p_0\, u\, (c_0) + p_1\, u\, (c_1) = k$, with k being a constant. In actuality, an infinite series of utility curves can be sketched out, offering ever-expanding pairings of "indifference" between the two consumption choices.

The budget constraint is: $p_0\, c_0 + p_1 c_1 = w$. (c_0, c_1) defines the set of claims for which expected utility is constant. The budget constraint can also be described as: $\Pi_0\, c_0 + \Pi_1\, c_1 = w$, as the probabilities are replaced by the pricing of claims, $\prod_s$. This leads to a equilibrium point constrained by asset pricing. The standard portfolio problem can now be stated as:

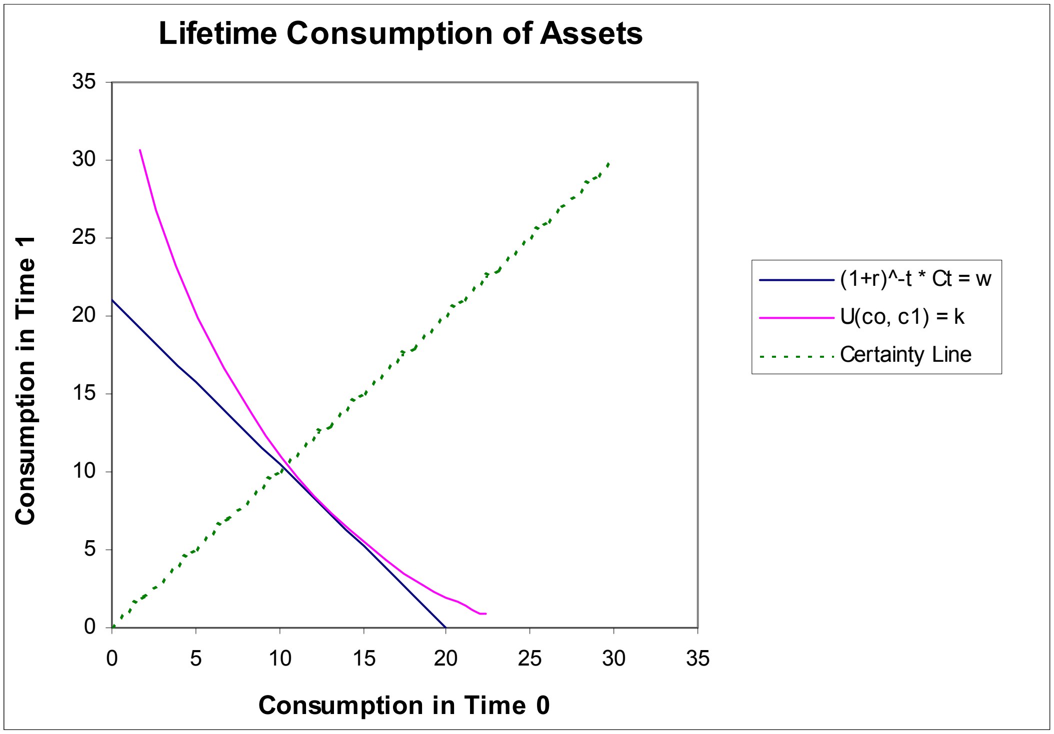
$$\max_{c0\ldots..cs} \Sigma^{S}_{s=1}\, p_s\, u\,(c_s) \quad \text{s.t.}\ \ \Sigma^{S}_{s=1} \prod_s c_s = w_0$$

The analysis can be extended to a multi-period setting in which individuals are able to defer gains and losses over several periods. The investor attempts to choose an optimal consumption path across time. This optimization is the same as the above consumption equilibrium of two claims in a single-period model, only now total consumption across the multiple periods is being optimized.

The lifetime consumption flows are $u\,(c_0, \ldots..c_{n-1})$, and the indifference curve is generally defined as $u\,(c_o, c_{n-1}) = k$. The budget constraint can be seen as: $\sum^{n-1}_{t=0}\ \Pi_t\, c_t \ <= w_0$.We assume an increasing and concave utility function. Optimal consumption is:

$$\max_c u(c_o\ldots c_{n-1}) \text{ s.t. } z_{t+1} = (1{+}r)\,(z_t + y_t - c_t)$$

The multi-period system so far can be visually described as:



In the above figure, consumption of a single asset occurs over two periods. The indifference curve is sketched in pink, while the blue line is the budget constraint. A power function with DARA and CRRA features was used. The preference for current consumption was set at 0.575 (and deferred consumption at .425). This resulted in a shift in the utility curve towards current consumption. If consumption would be the same in both time periods, then the tangency of the budget constraint and the indifference curves would occur at the certainty line, being the dashed line in the graph.

When time periods are made part of the process, the basic choice of immediate consumption versus deferred consumption becomes more evident than with portfolio theory associated with the special case.

3 (A) (2). Dynamic Methods

Bellman (1953) pioneered dynamic programming efforts. As applied to the investment process, investors can deliberately shift allocations in any one time period to maximize consumption across all time periods. This is a significant advantage for long-term investors. The objective is no longer to secure a high return in every single time period. The focus is now on an investment goal over an entire life cycle.

Backward induction can be used to solve dynamic problems, whereby the second period problem is solved first, for each possible outcome that could exist at the start of period 2. Thus, the second period is contingent upon state z, which was obtained in the first period.

$$u(z) = \max_{\alpha 1} u(z, \alpha_1)$$

where, α_1 is the optimal strategy in period 2. This transforms the static situation into a dynamic problem, where an investor can adjust strategies over time. Dynamic portfolio management becomes a sequence of static problems designed to maximize end period utility. Optimal exposure of the risky asset in the 1st period is determined by solving for the first order condition (FOC) of the above utility function. This relationship between exposure to the risky asset α_1 and wealth z, is: $u'(\alpha_1 z) = 0$. If the investor is more risk tolerant in the long-term than in the short-term, the optimal exposure to risk in the short-term will increase in order to maximize long-term utility. The agent will maximize utility long-term by:

$$v(z) = \max_{c0 \ldots cs-1} \sum^{S-1}_{s=0} p_s u(c_s) \quad \text{s.t.} \quad \sum^{S-1}_{s=0} \prod_s c_s = z$$

where, accumulated wealth is z. This equation is analytically the same as the baseline portfolio problem noted above, only now placed into a dynamic setting. The degree of absolute risk tolerance long-term is:

$$T_v(z) = -v'(z) / v''(z) = \sum^{S-1}_{s=0} \prod_s T(c^*_s)$$

The degree of absolute risk tolerance short-term will be:

$$T_u(z) = -u'(z) / u''(z) = \sum^{S-1}_{s=0} \prod_s T(c^*_s)$$

Where c* is the optimal solution to v (z), above. Allocating short-term risk over a lifetime of consumption can be calculated in the following equation.

$$v(z) = \max_c \sum^{n-1}_{t=0} p_t u(c_t) \quad \text{s.t.} \sum^{n-1}_{t=0} c_t = z + n y$$

where, z is a given wealth accumulated prior to t = 0; p_t is the discount factor associated with date t; and z + n y is the lifetime wealth. Optimal initial risk is determined by solving for $\max_{\alpha 0} E\, v(z(\alpha_0, x\sim)$.

Overall, the advantage of a long-term investment perspective comes from the agent being able to break-up the risks on lifetime consumption into smaller, discrete components of shorter holding periods. The agent is then able to adjust exposure to risky assets in these shorter time frames in order to maximize lifetime consumption. The absolute tolerance to risk on consumption over the agent's lifetime remains the same, but the degree of risk tolerance in shorter time frames will change to accommodate the overall goal of consumption maximization over an entire life.

3 (A) (3). The Predictability of Returns

Predictability in the return distribution is a huge factor in portfolio decision-making. Reichenstein & Dorsett (1995) found that bad periods of market behavior are *predictably* followed by good periods, and visa versa. The degree of serial correlation in equity returns is so strong that standard text-books, such as Bodie, Kane, and Marcus (2004), have even acknowledged the possibility of profit opportunities.

In the presence of return predictability, long-horizon investors can take more risks early in life Predictability exists when the realized return for each period, $x\sim_1$ is correlated to $x\sim_0$. $E\, x\sim_0 > 0$, with predictability of returns. The value function is:

$$u\,(z, x_0) = \max_\alpha E\,[\,(z + \alpha\, x\sim_1\, x\sim_1)^{(1-\gamma)} / ((1-\gamma) \mid x_0].$$

Where z is now used for wealth. The function will be separable for each period, and the first period problem can now be solved via backward induction. Gollier (2005) as to the baseline portfolio problem with:

$$v_{n-1}\,(z, s_{n-1}) = \max_{c1\ldots cs} \Sigma^S{}_{s=1}\, p_s\, u\,(c_s) \;\; \text{s.t.}\; \Sigma^S{}_{s=1} \prod_s (s_{n-1})\, c_s = z$$

where, the vector of prices in the last period depends upon states of nature s-1 that prevailed one period earlier. Return predictability will induce an agent to take more risk so long as coefficient of RRA, $\gamma > 1$. For log utility functions, constant allocations will still be optimal. The agent will not take more risk, even with predictability.

Predictability can have the same effect as DRRA when the level of optimal risk increases over time. To Gollier (2002) in particular, the choice of initial portfolio pricing risk is driven by marginal value of wealth, which in turns depends upon the future opportunity set. If predictability raises the marginal value of terminal wealth, then it has the same effect as a decline in relative risk aversion. The central question becomes the extent of change in the opportunity set on marginal wealth.

The predictability of stock returns across time horizons is now seen as only one of several types of predictability. Gollier (2002) categorized predictability into different forms, with varying impacts upon overall portfolio allocations. Predictability of long-term bonds is of a first order stochastic dominance (FSD). Predictability of a second order dominance (SSD) comes from mean reverting stocks and the stochastic volatility of returns. Three effects upon predictability have also identified: substitution, wealth, and precautionary. These effects may be offsetting to each other, depending upon the type of predictability and other investor level factors involved.

Where the return distribution is not perfectly known, the optimal strategy is affected. This type of parameter uncertainty will tend to make the agent more conservative than with a return structure than is completely known. This was shown in the above example with Barberis (2000), as well as in Gollier (2005).

3 (A) (4). Other Factors in the Opportunity Set

When expected utility constructs are used that are general in nature, numerous additional factors could be added to the process. Any of these factors could serve to reverse or reinforce the economic impacts of risk aversion and return distribution. See, Kaufhold (2007:1) for an extensive discussion of these factors.

Outlining some of these items, macro influences on asset pricing should be factored into the analysis. Interest rates, and investor's views on what constitutes the risk-free rate of return are important, especially when inflationary effects are considered. Background and uncertainty over an investor's environment is also a consideration. Estimation risk looms large with both the special and general cases. Liquidity constraints, initial wealth levels, and labor income shocks are all very important. Other factors include the incentive to save, work habits, the ability to learn, the frequency of withdrawal of assets, the existence of known future periods of withdrawal, existence of information, the incompleteness of the information.

Adding the full opportunity set to the dynamic process yields interesting results. Long-term investors possessing decreasing risk aversion and who can take advantage of return predictability will want to split their gains and losses into small, annual components parts over an entire lifetime of consumption. The solution to the dynamic portfolio problem then depends upon the existence and intensity of all factors of the opportunity set. For example, the effects of DARA and predictability could be limited or even completely eliminated by these other factors. Thus, the general case of utility should consider all of these various factors before a final portfolio solution is arrived at.

3 (A) (5). Diversification of Assets

From Eeckhoudt, et al (2001), assuming a concave utility function, marginal utility of wealth would be decreasing. Concavity of utility implies:

$$Eu\,(w + z) <= u\,(w + Ez)$$

Where, z is a lottery, Ez is the expected payoff of the lottery, and w is wealth. The welfare of the lottery will be less than the expected payoff of that lottery with certainty. The expected payoff will mathematically be the same as the lottery, but the utility of the lottery will now be less than the utility of the expected payoff. If x_1 and x_2 are independent and identical (IID) and have the same return probability, then:

$$\max_{\alpha} Eu\,(\alpha\, x_1 + (w - \alpha)\, x_2\,)$$

where, α is the amount invested in the first risky asset. Solving for the first order condition yields: $E(x_1 - x_2)\, u'(\alpha^* x_1 + (w - \alpha^*)\, x_2) = 0$. Because x_1 and x_2 are assumed to have the same return distributions, they can be added together:

$$E\, x_1\, u'\,(\tfrac{1}{2}\, w\, (x_1 + x_2) = E\, x_2\, u'\,(\tfrac{1}{2}\, w\, (x_1 + x_2))$$

Because $x\sim_1$ and $x_{\sim 2}$ are IID, they can be interchanged. Thus: $\alpha^* = \tfrac{1}{2}\, w$.

For two IID assets with the same return probability mass, expected utility will be maximized by each asset comprising one-half of the portfolio. All other portfolios will be inferior to this equally divided composition. No particular utility function has to be calculated in order to conclude that diversification maximizes utility of the investor. For assets with identical returns distributions, the only requirement necessary to prove the value of diversification is the existence of investor-level risk aversion.

3 (A) (6). Allocation of Assets

Continuing with the above thoughts, assume that two agents are both risk-averse and have independent possible occurrences, one that fails and the other that succeeds. Not only should each agent diversify the risks by developing separate choices, the agents could also share risks in each other's endeavors.

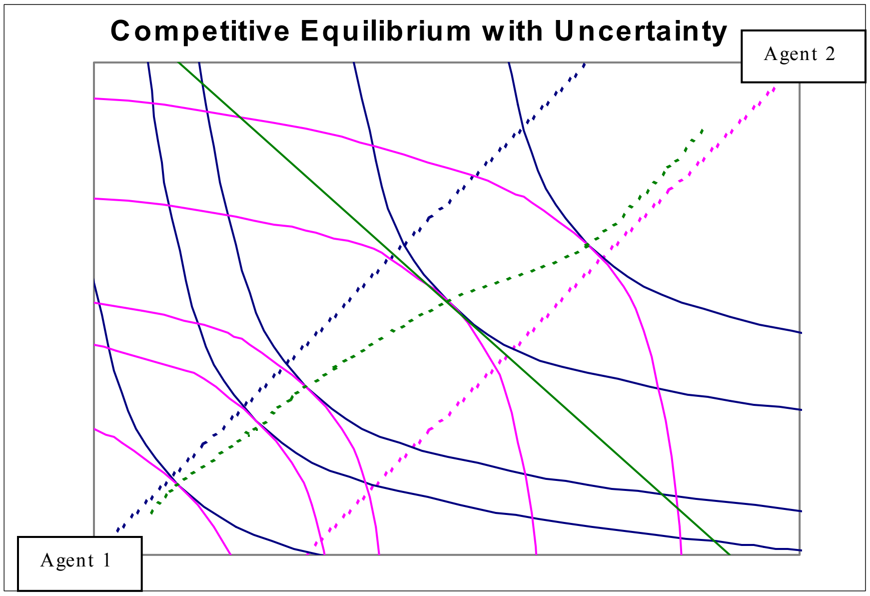
With two or more agents, risk sharing is Pareto-improving. If one agent has more risk aversion, then it would be Pareto-optimal to transfer some of the aggregate risk to the more risk-tolerant agent in exchange for a lump sum payment or risk premium. The equilibrium that describes this transfer of risk is known as the contract curve, since this line is based upon mutually agreed upon risk assumption in exchange for the risk premium. Generally, risk allocation would occur if aggregate consumption equals total wealth such that: $z\,(s) = \Sigma_i\, w_i\,(s)$ available in state: $\Sigma^n_{i=1}\, c_i = z\,(s)$. .

When macro-economic uncertainty is introduced, the possibility of fully insuring all risks between willing investors disappears. While full coverage of all risks by any individual agent will no longer be possible due to uncertainty, the risks can be shared and shifted between agents in the area of uncertainty, defined as the area between the two certainty lines of the graph, below. Assuming well-behaved markets with complete information, the sharing of risks will lead to elimination of all non-diversifiable risks. This gives rise to the "mutuality principle" whereby all diversifiable risks will be washed out by mutual arrangement of agents. Thus, the general case of utility proves the value of both allocation and diversification of assets.

The "contract curve" will be the Pareto-efficient allocation of risk that can raise EU of one agent without reducing EU of the other agent. The contract curve in the following graph is the dashed green line. The Pareto efficient allocation will be:

$$\text{Max}_{c(1)\ldots c(n)}\, \Sigma^n_{i=1}\, \lambda_i\, E\, u_i\,(c_i\,(s^{\sim})\ \text{subject to}\ \Sigma^n_{i=1}\, c_i = z\,(s).$$

λ_I is a vector of possible scalars. The optimal solution can be solved state by state, as a sequence of simpler problems. The decision variables will be concave as to their functions.



The non-diversifiable macroeconomic risk that was previously assumed appears as the range between the two diagonal certainty lines of the Edgeworth box, above. The greater the risk tolerance of an agent relative to the aggregate risk tolerance of the group, the greater the risk that particular agent will bear. The Pareto-efficient solutions will be within the diagonal lines, with agreement occurring there between agents.

This graph portrays in general terms similar optimality as the above figure noting the tangency of the efficient frontier with the CAPM in the special case.

Decentralized pricing decisions can be used to allocate risks. Assume that each agent has securities in which to exchange risk. There will also be some uncertainty about state $s^{\sim}$ occurring at the end of each period, due to macro-economic risks. The agent *i* will find a portfolio c_i (.) of Arrow-Debreu securities that maximizes EU within a budget constraint.

$$\max_{c_i(.)} Eu_i (c_i (s^{\sim})), \text{ s.t. } E\, \pi (s^{\sim})\, c_i (s^{\sim}).$$

The portfolio cannot exceed the market value of the initial endowment. A market clearing condition exists whereby aggregate consumption in state *s* cannot exceed what is available in that state. In the absence of externalities and asymmetric information, a competitive allocation will be Pareto-optimal for risk-averse EU maximizers. Competitive markets will allocate the macroeconomic risk in a Pareto-efficient manner.

The contract curve of the Edgeworth box will be the competitive equilibrium, and a budget constraint will then define and limit the actual level of consumption by the agents. Since the agents will be risk sharing between the two certainty lines, they will have a higher EU than in autocracy. The exact position of the contract curve will depend upon the RRA of each agent, with risk tolerant agents taking more of the risk in the area of uncertainty.

The analysis can be extended to a multi-period dynamic model of n assets. Adding multiple time periods and numerous assets do not affect the properties noted above. Mutuality, macro risk, risk sharing are all still present.

4. Conclusion

This paper explores expected utility concepts applicable to investment and portfolio analysis. The special case of utility maximization, complete with its various assumptions, can be used as a starting part. Significant limitations exist within the special case however, namely the complete absence of liabilities, the inability to effectively vary and model risk aversion or the return distribution structures, and the inherently increasing risk aversion tendencies of the quadratic utility function. The special case is therefore not overly useful in handling the full opportunity set of economic issues facing investors. To some extent, the special case can be defended as returning the same optimal conditions as with first and second order deteriorating positions. Means-variance can also be somewhat extended to cover the forward liability streams through Asset-Liability Modeling, Monte Carlo simulations, and other liability-based efforts.

To cognitively model investor behavior and to more completely state a capable theory of investment processes however, a generalized notion of utility is necessary. Optimality, equilibrium, the standard portfolio problem, diversification, and allocation are present with both the special and general cases of utility. Whereas the special case can present that optimality in only a very restrictive manner, the general case states equilibrium conditions with varying assumptions and in all types of manners. For instance, general expected utility can generate the same time invariant optimal allocation as does the special case by simply assuming power utility, CRRA, and IID. It can also emulate optimality of a quadratic by assuming IRRA. But it can go far beyond these results, and produce increasing or decreasing optimal allocations by adjusting relative risk and return distribution assumptions, as well as by a consideration of the numerous other factors of the opportunity set. The special case is simply incapable of this flexibility and level of comprehensiveness.

Further, by deliberately adjusting short-term allocations to maximize life-time consumption, the general state of expected utility can produce a different set of optimal conditions than with the special situation of normality. This represents a quantum jump over the special case with its unitary time horizon, and quite possibly could result in significant long-term economic benefits to investors who reference the general case of utility maximization.

References

K. J. Arrow, 1953; translated as "The Role of Securities in the Optimal Allocation of Risk-Bearing," *Review of Economic Studies,* 31 (1964): 91-96;

Nicholas Barberis, "Investing in the Long Run When Returns are Predictable", *The Journal of Finance,* 2000, no. 1, pp. 225-265.

Richard Bellman, *An Introduction to the Theory of Dynamic Programming,* Rand Corporation (1953).

Daniel Bernoulli, "Exposition of a New Theory on the Measurement of Risk," 1738; originally in Latin; translated into English by Louise Sommer, *Econometrica,* 22: (1954): 23-36.

Zvi Bodie, Alex Kane, and Alan J. Marcus, *Essentials of Investments,* McGraw-Hill Irwin, 2004.

Bruce Cable and Mike Rust, "The Funded Status of a Multiemployer Pension Plan: What it is and What is Appropriate?", International Foundation of Employee Benefit Plans, Annual Conference (2005)

John Y. Campbell and Luis M. Viceira, *Strategic Asset Allocation,* Oxford University Press, NY (2002; reprinted 2003).

G. Debreu, *The Theory of Value,* Wiley and Son, (1959)

Louis Eeckhoudt, Christian Gollier, and Harris Schlesinger, *Economic and Financial Decisions Under Risk,* Princeton University Press, 2005.

Edwin J. Elton, Martin J. Gruber, Stephen J. Brown, and William N. Goetzmann, *Modern Portfolio Theory and Investment Analysis,* John Wiley & Sons, 6th ed., 2003.

Eugene Fama, *Foundations of Finance,* 1976.

Christian Gollier, "Optimal Portfolio Management for Individual Pension Plans", CESIFO Working Paper No. 1394, Feb. 2005.

Christian Gollier, "Time Diversification, Liquidity Constraints, and Decreasing Aversion to Risk on Wealth", *Journal of Monetary Economics,* Vol. 49, No. 7, October 2002, p. 1493-1459.

Christian Gollier, *Economics of Risk and Time,* MIT Press, 2001.

Ibbotson Associates, *SBBI Yearbook*, 2005 edition.

Kevin Kaufhold, “Time-Referenced Investment Policies”, *Journal of Financial Planning,* April 2008.

Kevin Kaufhold, “Developing Time Horizons for Use in Portfolio Analysis”, IFEBP Benefit and Compensation Digest, March, 2007, WebExclusive

Kevin Kaufhold, “General Utility Concepts Applicable to Investments”, working paper 2007:1.

John Lintner, “Security Prices, Risk and Material Gains from Diversification”, *Journal of Finance* 20, no. 4 (December 1965): 587-615.

Harry Markowitz, *Portfolio Selection: Efficient Diversification of Investment,* John Wiley & Son, 1959.

Harry Markowitz, “Portfolio Selection,” *Journal of Finance* 7, no.1 (March 1952): 77-91.

Robert Merton, “Lifetime Portfolio Selection Under Uncertainty: The Continuous Time Case”, *Review of Economics and Statistics,* vol. 51, no. 3 (1969), pp.247-257.

Jan Mossin, “Equilibrium in a Capital Asset Market”, *Econometrica* 34, no. 4 (October 1966): 768-783.

J. von Neuman and O. Morgenstern, *Theory of Games and Economic Behavior,* Princeton University Press, 1948.

Reichenstein & Dorsett, “Time Diversification Revisited”, AIMR publication, 1995.

A.D. Roy, “Safety-First and the Holding of Assets,” *Econometrica,* 20 (1952): 431-439.

Mark T. Ruloff ,“Assessing Your Plan’s Asset Liabilities,” International Foundation of Employee Benefit Plans, Annual Conference (2006)

Paul Samuelson, “Lifetime Portfolio Selection by Dynamic Stochastic Programming”, *Review of Economic and Statistics,* vol. 51, no. 3 (August 1969): 247-257.

William F. Sharpe, “Capital Assets: A Theory of Market Equilibrium under Conditions of Risk”, *Journal of Finance* 19, no. 3 (September 1964): 425-442.

David Stangeland and Harry J. Turtle, “Time Diversification: Fact or Fallacy”, *Journal of Financial Education,* Fall 1999, at 1-13.